Dynamic Capacity Management for Deferred Surgeries

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Spotlight Talk

Joint work with Eojin Han, Kristian Singh, and Omid Nohadani

SMU LYLE SCHOOL OF ENGINEERING

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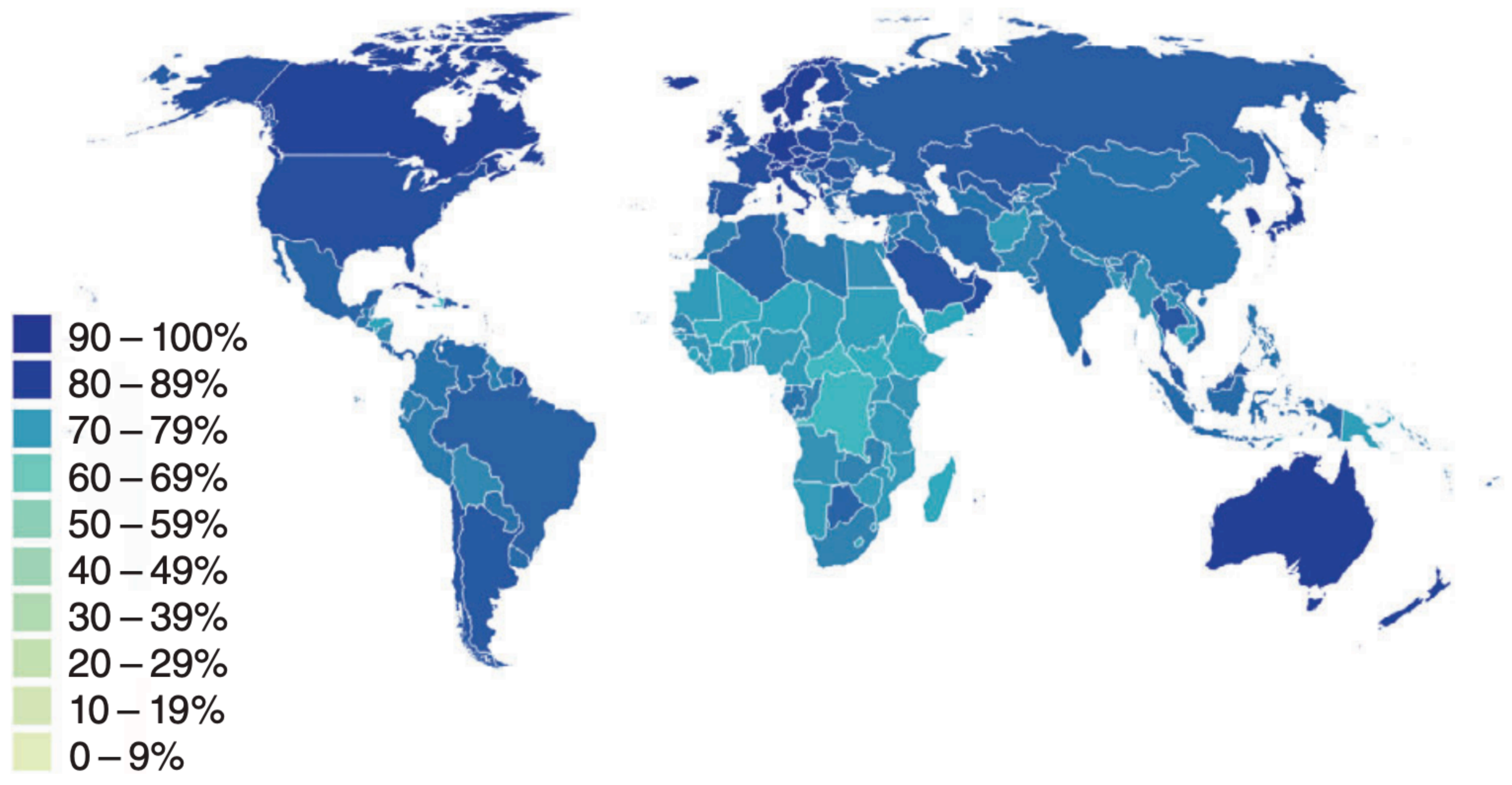
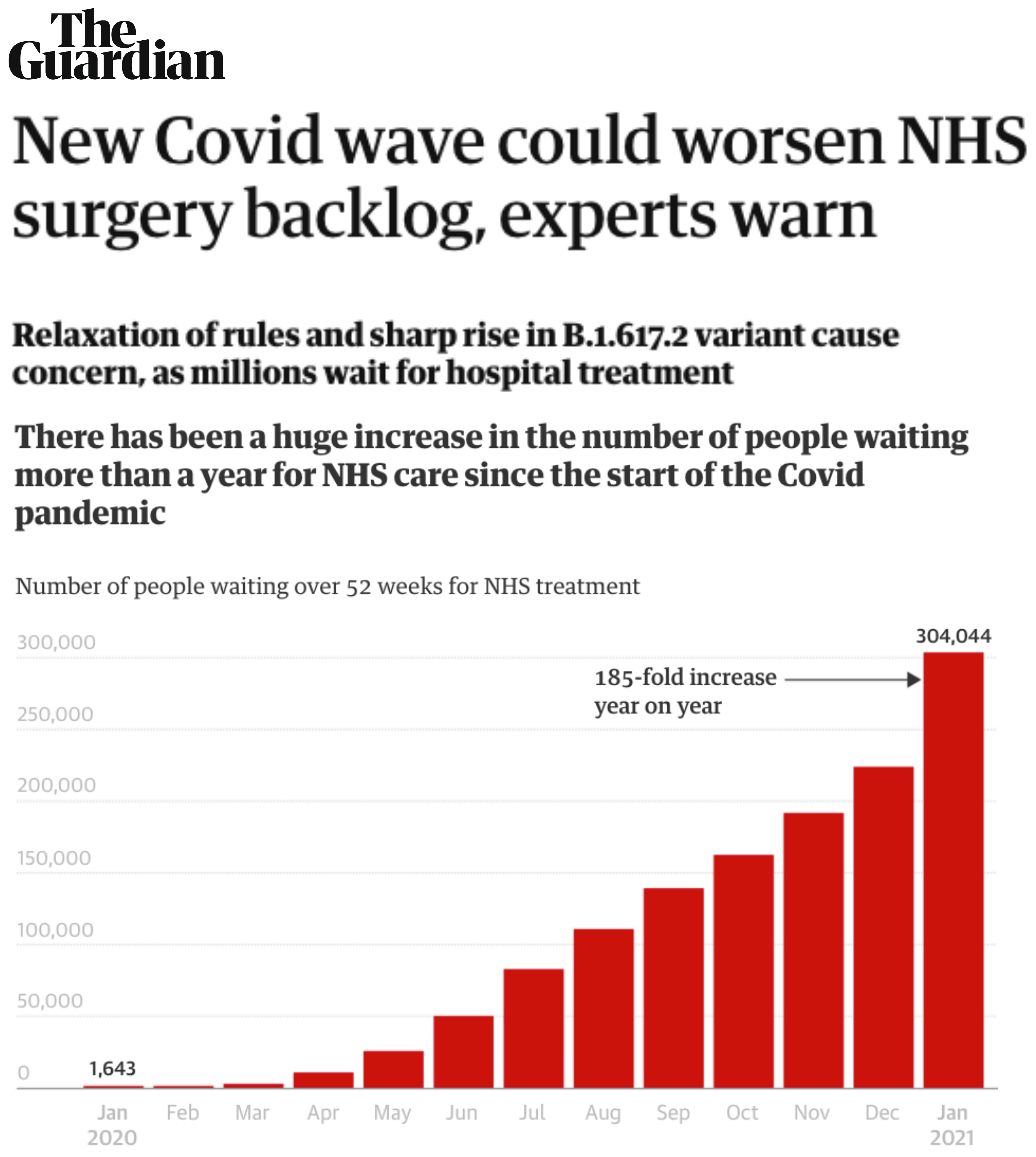
Benefits Science™ Technologies

Deferred Surgeries due to COVID-19

- 12-week cancellation rates of surgery for benign disease (March to May 2020)



Source: COVIDSurg Collaborative (2020) Elective surgery cancellations due to the COVID-19 pandemic: global predictive modeling to inform surgical recovery plans. British Journal of Surgery, 107(11): 1440-1449.


The Guardian

New Covid wave could worsen NHS surgery backlog, experts warn

Relaxation of rules and sharp rise in B.1.617.2 variant cause concern, as millions wait for hospital treatment

There has been a huge increase in the number of people waiting more than a year for NHS care since the start of the Covid pandemic

Number of people waiting over 52 weeks for NHS treatment

Source: D. Campbell. ‘A truly frightening backlog’: ex-NHS chief warns of delays in vital care. The Guardian, April 2, 2021 / N. Davis and D. Campbell. New Covid wave could worsen NHS surgery backlog, experts warn. The Guardian, May 20, 2021.

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- Potentially, worst health care outcomes for patients due to delayed treatment
- Increased financial costs for hospitals and insurers due to worsened diseases
- Significant financial loss for hospitals
  - Average monthly loss of revenue of the U.S. hospitals is $50.7 billion for March-June 2020 (Meredith et al. 2020).
  - Elective surgeries account for 43% of gross revenue of the U.S. hospitals (Tonna et al. 2020).

Source: Meredith, High, and Freischlag (2020) Preserving elective surgeries in the COVID-19 pandemic and the future. JAMA 324(17):1725-1726. Tonna, Hanson, Cohan, McCrum, Horns, Brooke, Das, Kelly, Campbell, and Hotaling (2020) Balancing revenue generation with capacity generation: case distribution, financial impact and hospital capacity changes from cancelling or resuming elective surgeries in the US during COVID-19. BMC Health Services Research 20(1):1-7.

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➡ **Silver Bullet**: An *optimization-based methodology* to dynamically manage surgical capacity for deferred surgeries, while balancing the profit with service requirements.

Problem Set-up

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Expansion Decisions

- $\mathbf{C}_B = (C_{B,1}, \cdots, C_{B,t})$
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Surgeries and Deferrals

- $\mathbf{u}_t = (u_t^{(-L)}, \cdots, u_t^{(t)})$: deferred surgeries.
  - $u_t^{(\tau)}$: surgeries scheduled at τ and carried out at t.
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Expansion constraints

Dynamic Programming Formulation

- Cost at time t :

$$H_t(C_{B,t}, C_t, \mathbf{u}_t, \mathbf{x}_t, \mathbf{w}_t) := b_{B,t}(\hat{C}_t + C_{B,t}) + b_t C_t + c_t \sum_{\tau=-L}^{t} x_t^{(\tau)} + \sum_{\tau=-L}^{t} p_{t-\tau}(u_t^{(\tau)} - x_t^{(\tau)} - w_t^{(\tau)}) + \sum_{\tau=-L}^{t} f_{t-\tau} w_t^{(\tau)}$$

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- Dynamic programming (DP) model:

$$\min_{\mathbf{C}_B, C_1} \mathbb{E}_{d_1}\left[\min_{\mathbf{x}_1} \mathbb{E}_{\mathbf{w}_1}\left[H_1(\cdot) + \min_{C_2} \mathbb{E}_{d_2}\left[\min_{\mathbf{x}_2} \mathbb{E}_{\mathbf{w}_2}\left[H_2(\cdot) + \cdots + \min_{C_T} \mathbb{E}_{d_T}\left[\min_{\mathbf{x}_T} \mathbb{E}_{\mathbf{w}_T}\left[H_T(\cdot)\right]\right] \cdots \right]\right]\right]\right]$$

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- Challenges

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- Challenges
  - Lack of distributional information
  - Model difficult to solve

Demand and Departure Uncertainty

- Departing patients $\mathbf{w}_t$ depends on $\mathbf{u}_t$ and $\mathbf{x}_t$:

$$\mathbf{w}_t \leq \mathbf{u}_t - \mathbf{x}_t \text{ almost surely.}$$

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- Now $\mathbf{u}_t$ is described via **multilinear** functions of θ_t , d_t , and $\mathbf{x}_t$ as

$$u_t^{(\tau)} = \left(\prod_{k=\max(\tau,1)}^{t-1} \theta_k \right) d_\tau - \sum_{t'=\max(\tau,1)}^{t-1} \left(\prod_{k=t'}^{t-1} \right) x_{t'}^{(\tau)} \qquad \forall \tau = -L, \cdots, t \ \forall t \in [T] \, .$$

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- We take a (distributionally) robust optimization approach to address this multilinearity.

Outline of Methods

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  - Uncertainties are described via polyhedral and box sets.
  - Decisions are made to minimize the worst-case cost.
  - Introduce the *tree of uncertainty products* and leverage McCormick relaxations to handle multilinear uncertainty.

$$\mathcal{U}_w(\mathbf{u}_T, \mathbf{x}_T) =$$

$$\{\mathbf{w}_t \mid \mathbf{w}_t = \rho_t(\mathbf{u}_t - x_t),\ \boldsymbol{\rho} \in \mathcal{U}_\rho\}$$

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  - Uncertainties are described via unknown distributions, which are described via sets.
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$$F \in \mathcal{M}_+ \text{ s.t.}$$
$$\mathbf{P}_F\left(\xi_t \in [\underline{\xi}_t\ ,\ \overline{\xi}_t]\right) = 1$$
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- Numerical Experiments

Overall Problem

- Overall problem:

$$\min_{C_t(\cdot),\mathbf{x}_t(\cdot)} \max_{\theta_{[T]}\in\Theta, d_{[T]}\in\mathcal{U}} \sum_{t\in[T]} G_t\left(C_t(\theta_{[t-1]}, d_{[t-1]}), \mathbf{x}_{[t]}(\theta_{[t-1]}, d_{[t]}), \theta_{[t]}, d_{[t]}\right)$$

$$\text{s.t.} \sum_{t'=\max(\tau,1)}^{t}\left(\prod_{k=t'}^{t-1}\theta_k\right) x_{t'}^{(\tau)}(\theta_{[t'-1]}, d_{[t']}) \leq \left(\prod_{k=\max(\tau,1)}^{t-1}\theta_k\right) d_\tau \quad \forall\theta_{[T]}\in\Theta, d_{[T]}\in\mathcal{U}, \tau\in[-L:t], t\in[T]$$

$$\sum_{\tau\in[-L:t]} x_t^{(\tau)}(\theta_{[t-1]}, d_{[t]}) \leq \hat{C}_t + C_{B,t} + C_t(\theta_{[t-1]}, d_{[t-1]}) \quad \forall\theta_{[T]}\in\Theta, d_{[T]}\in\mathcal{U}, t\in[T]$$

$$\mathbf{x}_t(\theta_{[t-1]}, d_{[t]}) \in \mathbb{R}_+^{t+L} \quad \forall\theta_{[T]}\in\Theta, d_{[T]}\in\mathcal{U}, t\in[T]$$

$$(\mathbf{C}_B, C_1, C_2(\theta_1, d_1), \cdots, C_T(\theta_{[T-1]}, d_{[T-1]})) \in \mathcal{C} \quad \forall\theta_{[T]}\in\Theta, d_{[T]}\in\mathcal{U},$$

where $G_t(C_t, \mathbf{x}_{[t]}, \theta_{[t]}, d_{[t]}) :=$

$$b_{B,t}(\hat{C}_t + C_{B,t}) + b_t C_t + \sum_{\tau=-L}^{t} c_t x_t^{(\tau)} + \sum_{\tau=-L}^{t} f_{t-\tau}\left[\left(\prod_{k=\max(\tau,1)}^{t-1}\theta_k\right) d_\tau - \sum_{t'=\max(\tau,1)}^{t}\left(\prod_{k=t'}^{t-1}\theta_k\right) x_{t'}^{(\tau)}\right]$$

$$+ \sum_{\tau=-L}^{t}(p_{t-\tau} - f_{t-\tau})\left[\left(\prod_{k=\max(\tau,1)}^{t}\theta_k\right) d_\tau - \sum_{t'=\max(\tau,1)}^{t}\left(\prod_{k=t'}^{t}\theta_k\right) x_{t'}^{(\tau)}\right].$$

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Can be $\mathbb{E}$ (stochastic), or $\sup \mathbb{E}$ (distributionally robust)

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$$\begin{aligned}
\text{s.t.} \quad & \sum_{t'=\max(\tau,1)}^{t} \left(\prod_{k=t'}^{t-1} \theta_k\right) x_{t'}^{(\tau)}(\theta_{[t'-1]}, d_{[t']}) \le \left(\prod_{k=\max(\tau,1)}^{t-1} \theta_k\right) d_\tau && \forall \theta_{[T]} \in \Theta, d_{[T]} \in \mathcal{U}, \tau \in [-L:t], t \in [T] \\
& \sum_{\tau\in[-L:t]} x_t^{(\tau)}(\theta_{[t-1]}, d_{[t]}) \le \hat{C}_t + C_{B,t} + C_t(\theta_{[t-1]}, d_{[t-1]}) && \forall \theta_{[T]} \in \Theta, d_{[T]} \in \mathcal{U}, t \in [T] \\
& \mathbf{x}_t(\theta_{[t-1]}, d_{[t]}) \in \mathbb{R}_+^{t+L} && \forall \theta_{[T]} \in \Theta, d_{[T]} \in \mathcal{U}, t \in [T] \\
& (\mathbf{C}_B, C_1, C_2(\theta_1, d_1), \cdots, C_T(\theta_{[T-1]}, d_{[T-1]})) \in \mathcal{C} && \forall \theta_{[T]} \in \Theta, d_{[T]} \in \mathcal{U},
\end{aligned}$$

where $G_t(C_t, \mathbf{x}_{[t]}, \theta_{[t]}, d_{[t]}) :=$

$$b_{B,t}(\hat{C}_t + C_{B,t}) + b_t C_t + \sum_{\tau=-L}^{t} c_t x_t^{(\tau)} + \sum_{\tau=-L}^{t} f_{t-\tau} \left[\left(\prod_{k=\max(\tau,1)}^{t-1} \theta_k\right) d_\tau - \sum_{t'=\max(\tau,1)}^{t} \left(\prod_{k=t'}^{t-1} \theta_k\right) x_{t'}^{(\tau)}\right]$$

$$+ \sum_{\tau=-L}^{t} (p_{t-\tau} - f_{t-\tau}) \left[\left(\prod_{k=\max(\tau,1)}^{t} \theta_k\right) d_\tau - \sum_{t'=\max(\tau,1)}^{t} \left(\prod_{k=t'}^{t} \theta_k\right) x_{t'}^{(\tau)}\right].$$

Overall Problem

- Overall problem:

Can be $\mathbb{E}$ (stochastic), or $\sup \mathbb{E}$ (distributionally robust)

$$\min_{C_t(\cdot),\mathbf{x}_t(\cdot)} \max_{\theta_{[T]}\in\Theta, d_{[T]}\in\mathcal{U}} \sum_{t\in[T]} G_t\left(C_t(\theta_{[t-1]}, d_{[t-1]}), \mathbf{x}_{[t]}(\theta_{[t-1]}, d_{[t]}), \theta_{[t]}, d_{[t]}\right)$$

$$\text{s.t.} \sum_{t'=\max(\tau,1)}^{t} \left(\prod_{k=t'}^{t-1} \theta_k\right) x_{t'}^{(\tau)}(\theta_{[t'-1]}, d_{[t']}) \le \left(\prod_{k=\max(\tau,1)}^{t-1} \theta_k\right) d_\tau \quad \forall \theta_{[T]} \in \Theta, d_{[T]} \in \mathcal{U}, \tau \in [-L:t], t \in [T]$$

$$\sum_{\tau\in[-L:t]} x_t^{(\tau)}(\theta_{[t-1]}, d_{[t]}) \le \hat{C}_t + C_{B,t} + C_t(\theta_{[t-1]}, d_{[t-1]}) \quad \forall \theta_{[T]} \in \Theta, d_{[T]} \in \mathcal{U}, t \in [T]$$

$$\mathbf{x}_t(\theta_{[t-1]}, d_{[t]}) \in \mathbb{R}_+^{t+L} \quad \forall \theta_{[T]} \in \Theta, d_{[T]} \in \mathcal{U}, t \in [T]$$

$$(\mathbf{C}_B, C_1, C_2(\theta_1, d_1), \cdots, C_T(\theta_{[T-1]}, d_{[T-1]})) \in \mathcal{C} \quad \forall \theta_{[T]} \in \Theta, d_{[T]} \in \mathcal{U},$$

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$$+ \sum_{\tau=-L}^{t} (p_{t-\tau} - f_{t-\tau})\left[\left(\prod_{k=\max(\tau,1)}^{t} \theta_k\right) d_\tau - \sum_{t'=\max(\tau,1)}^{t} \left(\prod_{k=t'}^{t} \theta_k\right) x_{t'}^{(\tau)}\right].$$

Multilinear uncertainty

Tree of Uncertainty Products

Tree of Uncertainty Products

- The problem consists of uncertain terms of the form

$$\sum_{k\in[K]} q^{(k)}\xi_k + \sum_{n\in[N]} q_g^{(n)} \prod_{i\in S_n} \xi_i \geq q_0 \quad \forall \boldsymbol{\xi} \in \mathcal{U}$$

- This constraint involves sums of multilinear terms.

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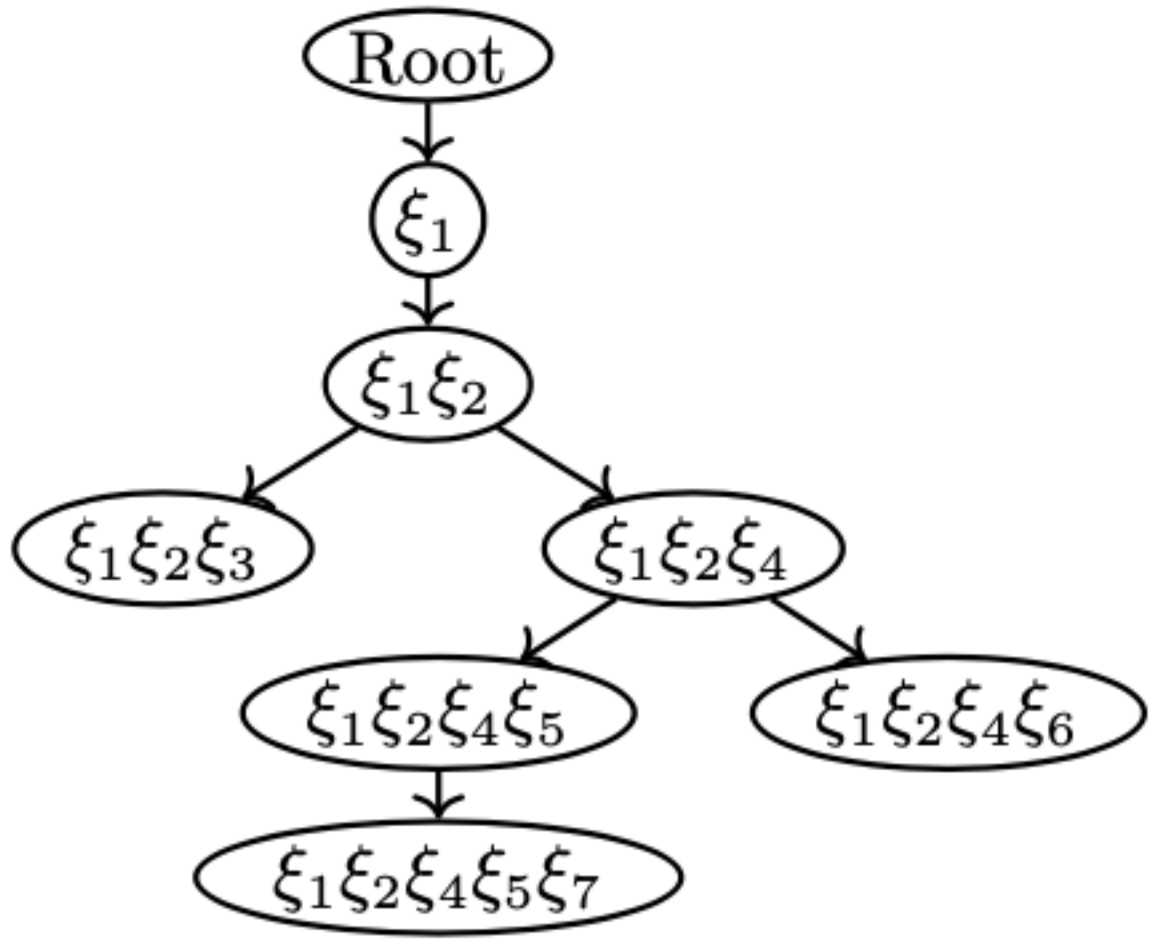


- This constraint involves sums of multilinear terms.

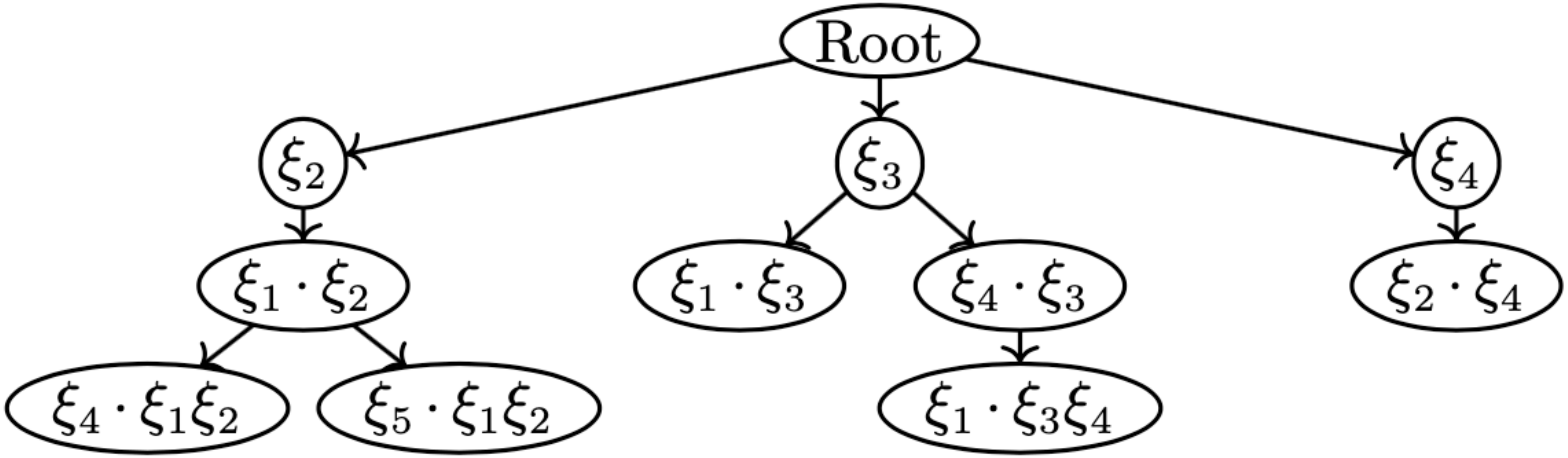
- We show that if
  - the multilinear products are in the form of leaves of tree, and
  - No two leaves without common ancestor share that uncertain component

then the constraint is equivalent to its McCormick relaxation

Tree of Uncertainty Products: Example

$\mathbf{z}^{(0)} = (0,0,0,0,0)$

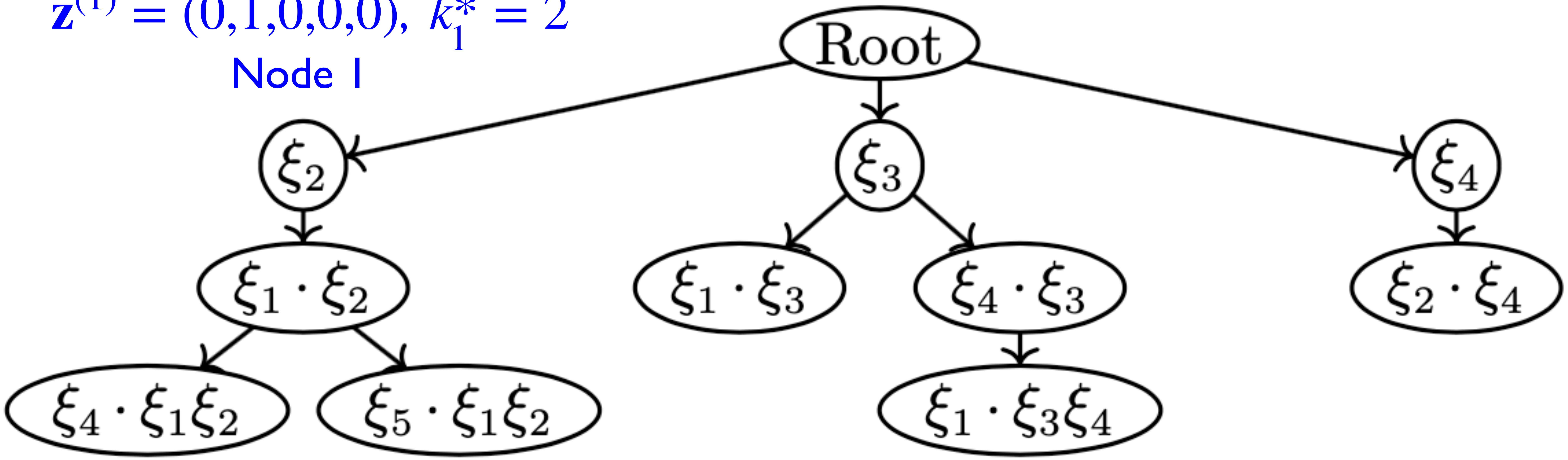
Root node (Node 0)



Tree of Uncertainty Products: Example

$\mathbf{z}^{(0)} = (0,0,0,0,0)$
Root node (Node 0)

$\mathbf{z}^{(1)} = (0,1,0,0,0),\ k_1^* = 2$
Node 1



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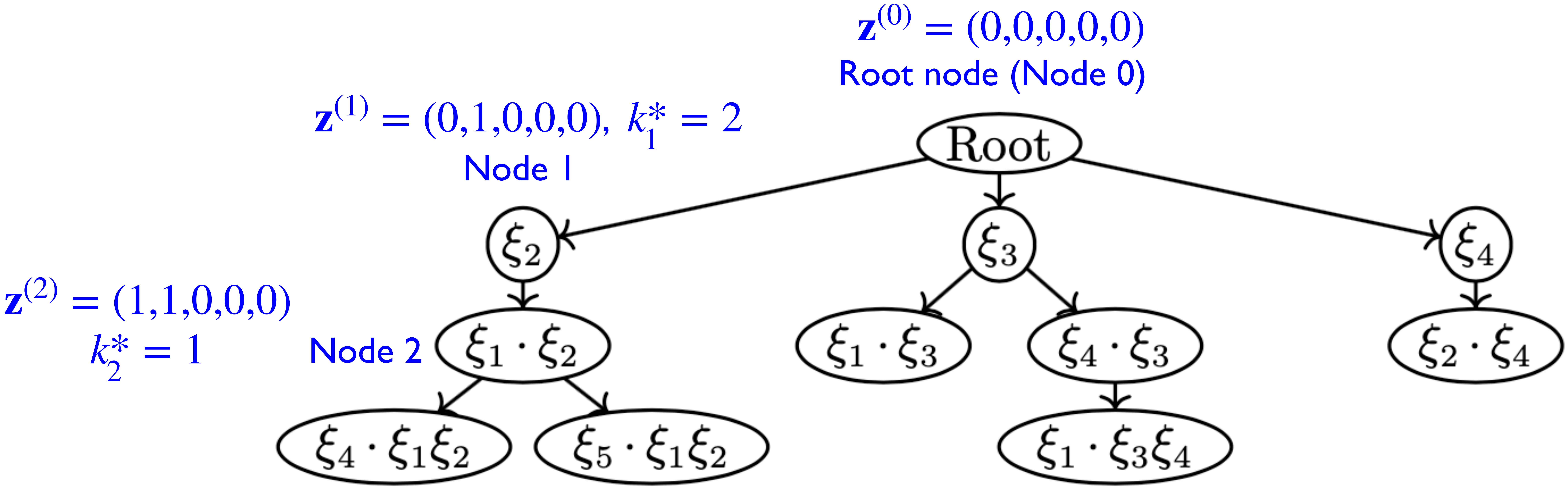

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Root node (Node 0)

$\mathbf{z}^{(1)} = (0,1,0,0,0),\ k_1^* = 2$

Node 1

$\mathbf{z}^{(2)} = (1,1,0,0,0)$

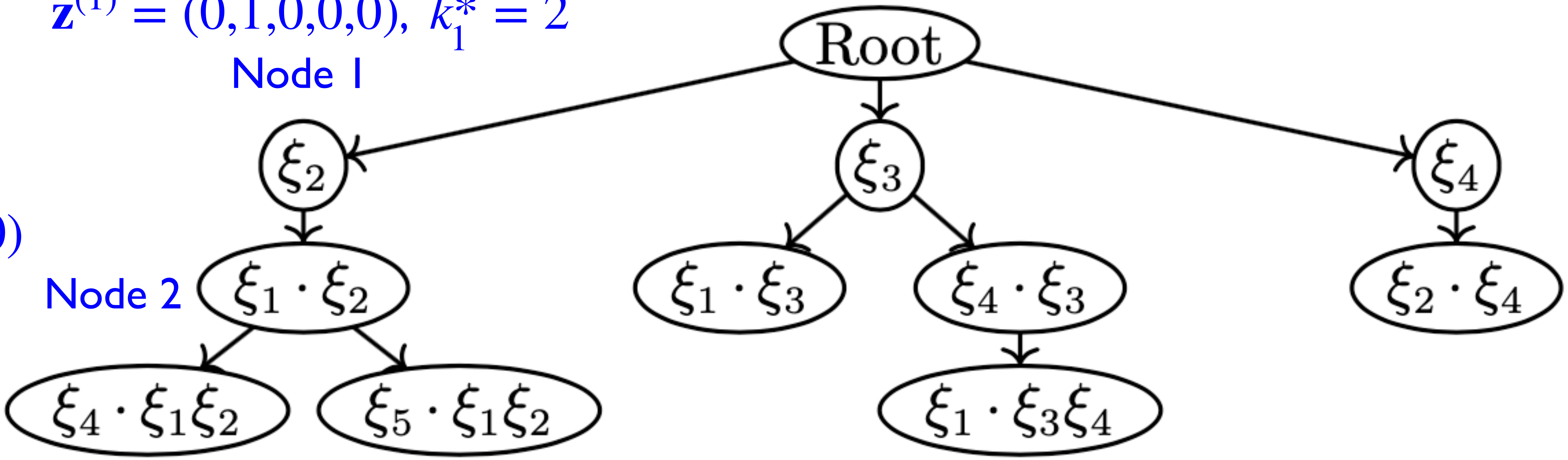
$k_2^* = 1$

Node 2

Node 3

$\mathbf{z}^{(3)} = (1,1,0,1,0)$

$k_3^* = 4$



Tree of Uncertainty Products: Example

$\mathbf{z}^{(0)} = (0,0,0,0,0)$
Root node (Node 0)

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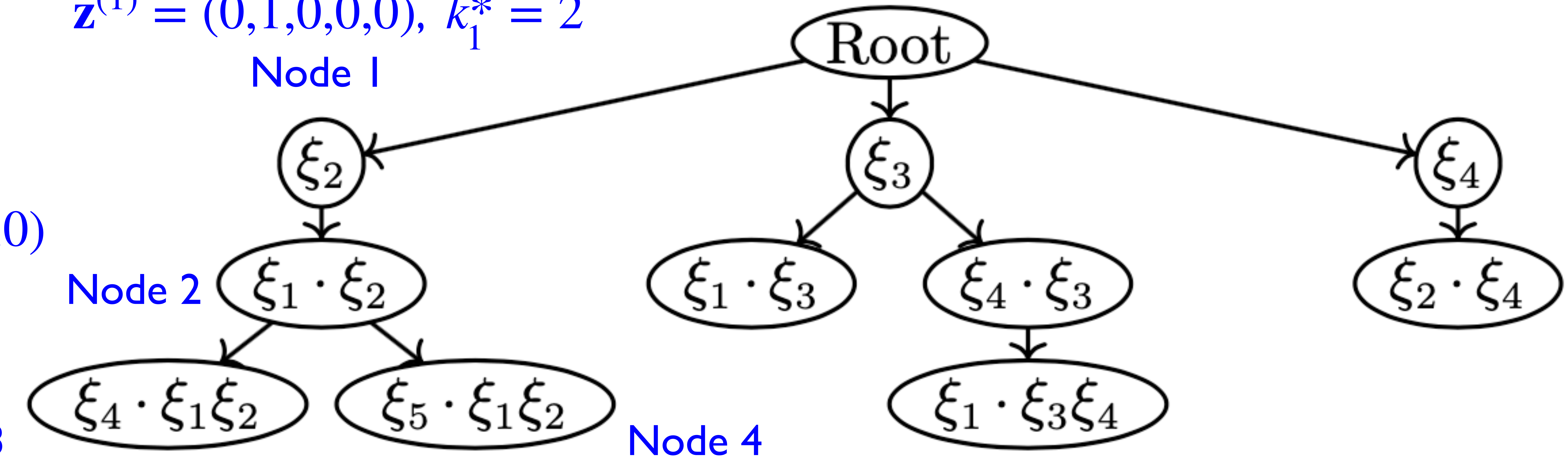
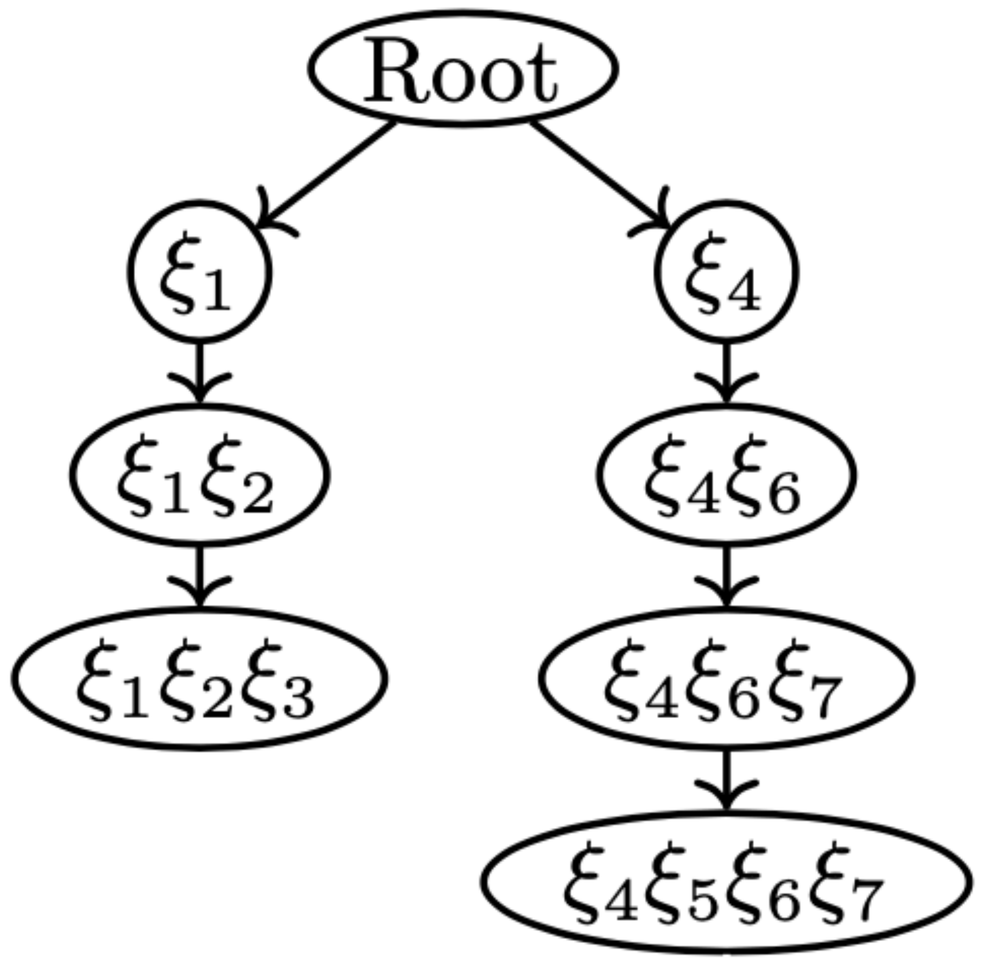


$\mathbf{z}^{(2)} = (1,1,0,0,0)$
$k_2^* = 1$
Node 2

Node 3
$\mathbf{z}^{(3)} = (1,1,0,1,0)$
$k_3^* = 4$

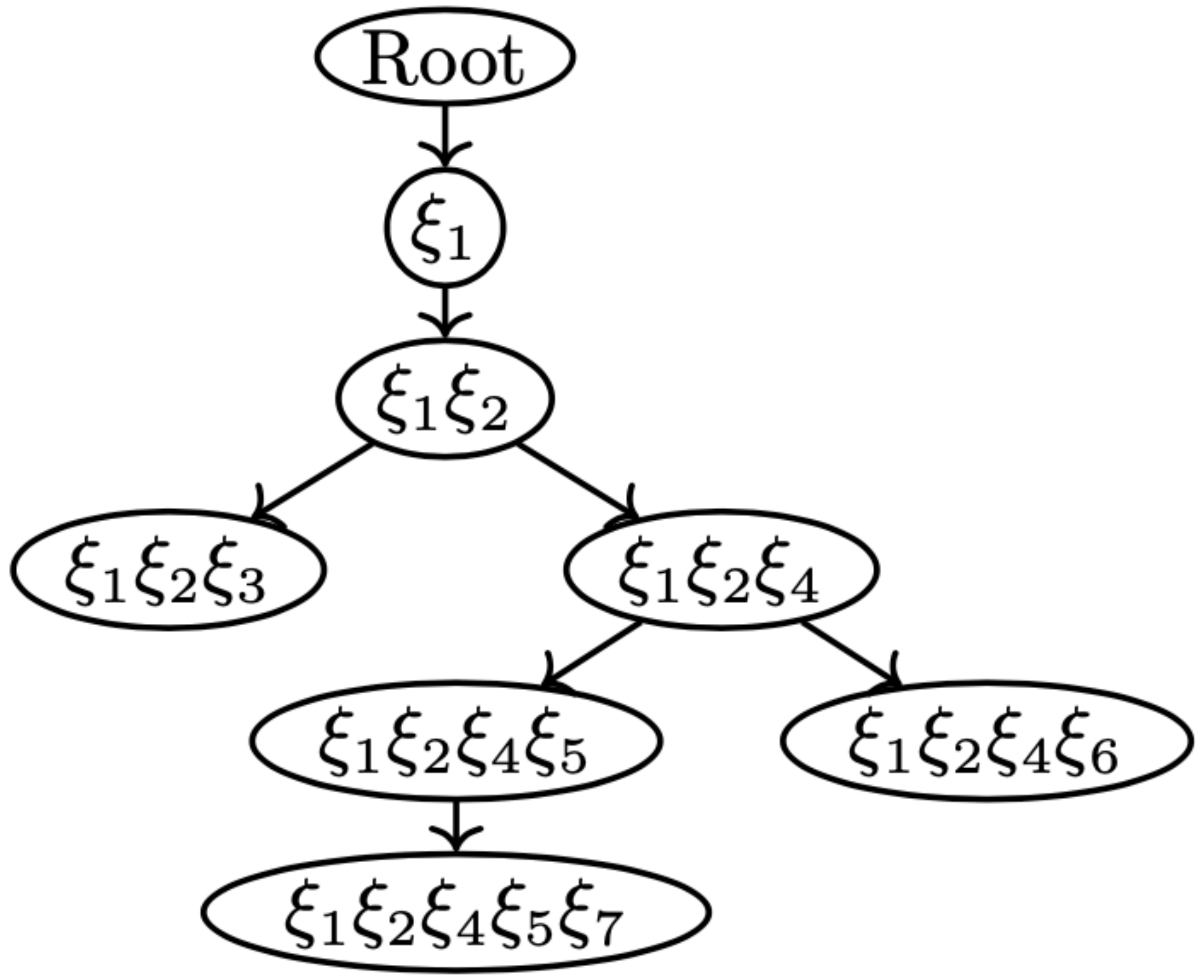
Node 4
$\mathbf{z}^{(4)} = (1,1,0,0,1)$
$k_4^* = 5$

Tree of Uncertainty Products



Exact

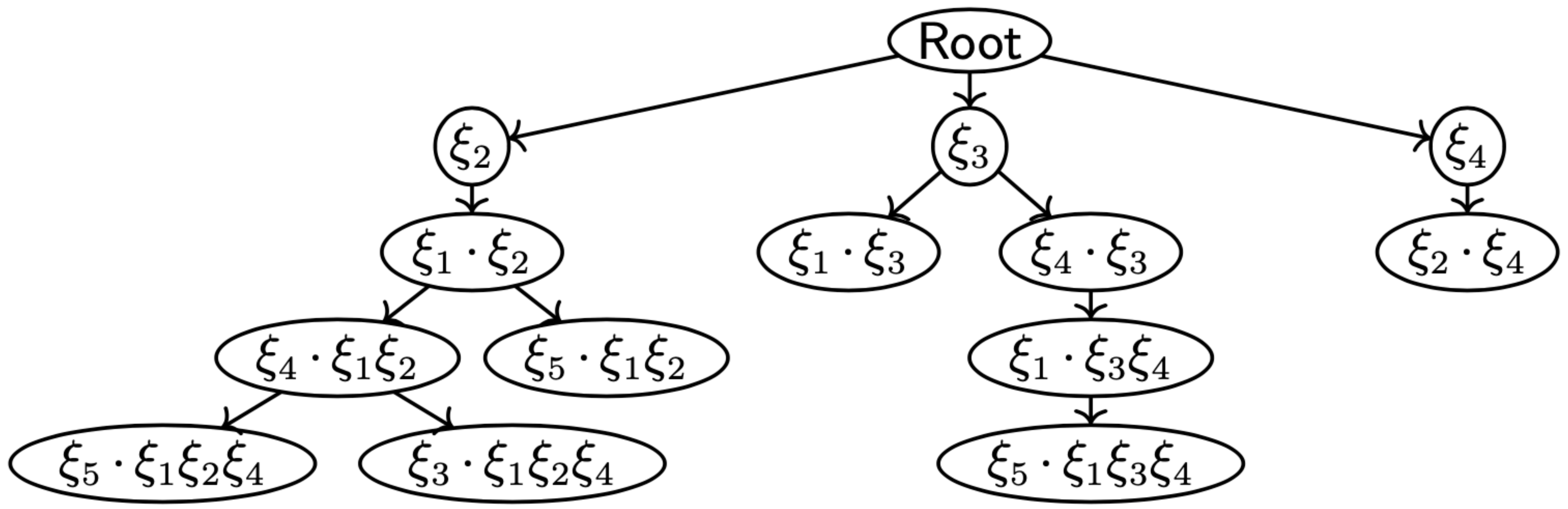
No shared ξ



Exact

No shared ξ

except from ancestor



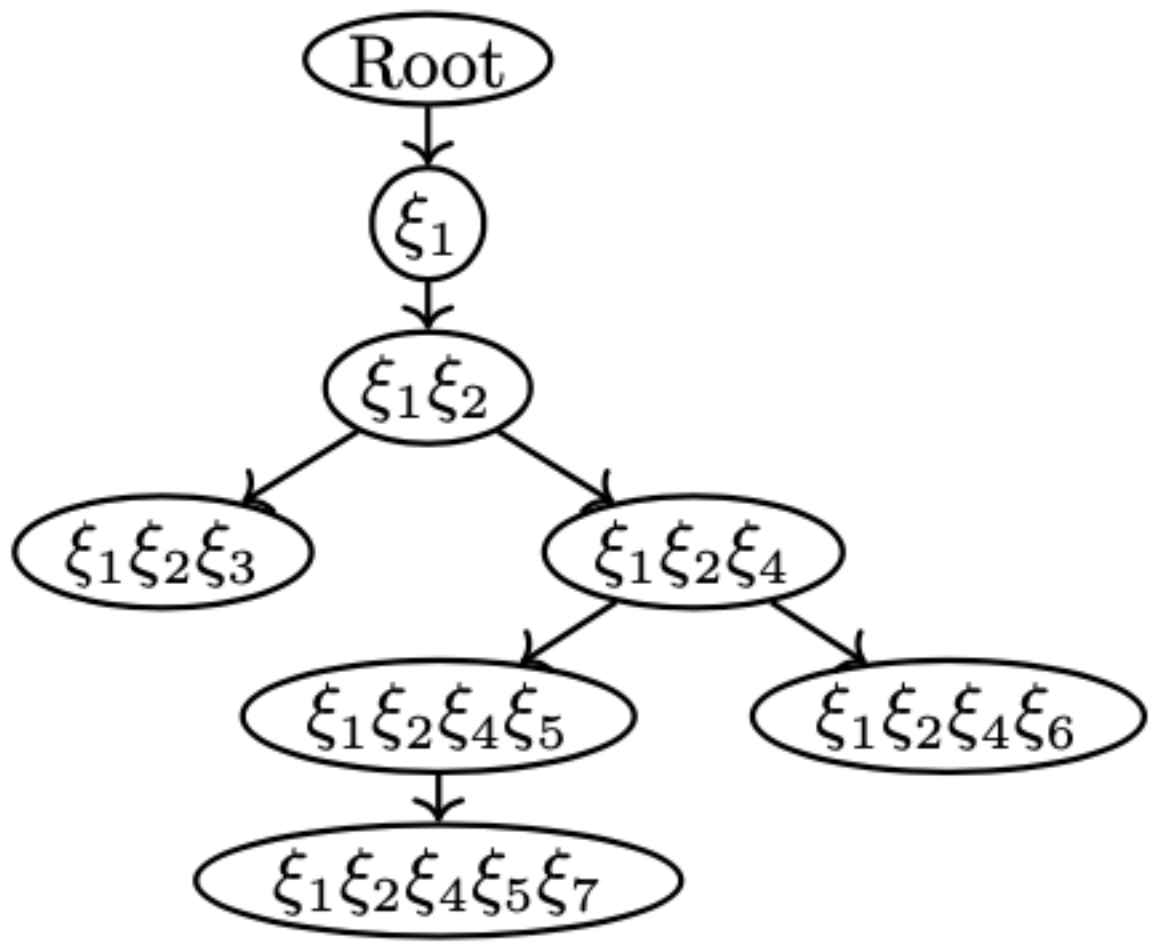
Not Exact

Shared ξ

Tree of Uncertainty Products

Tree of Uncertainty Products

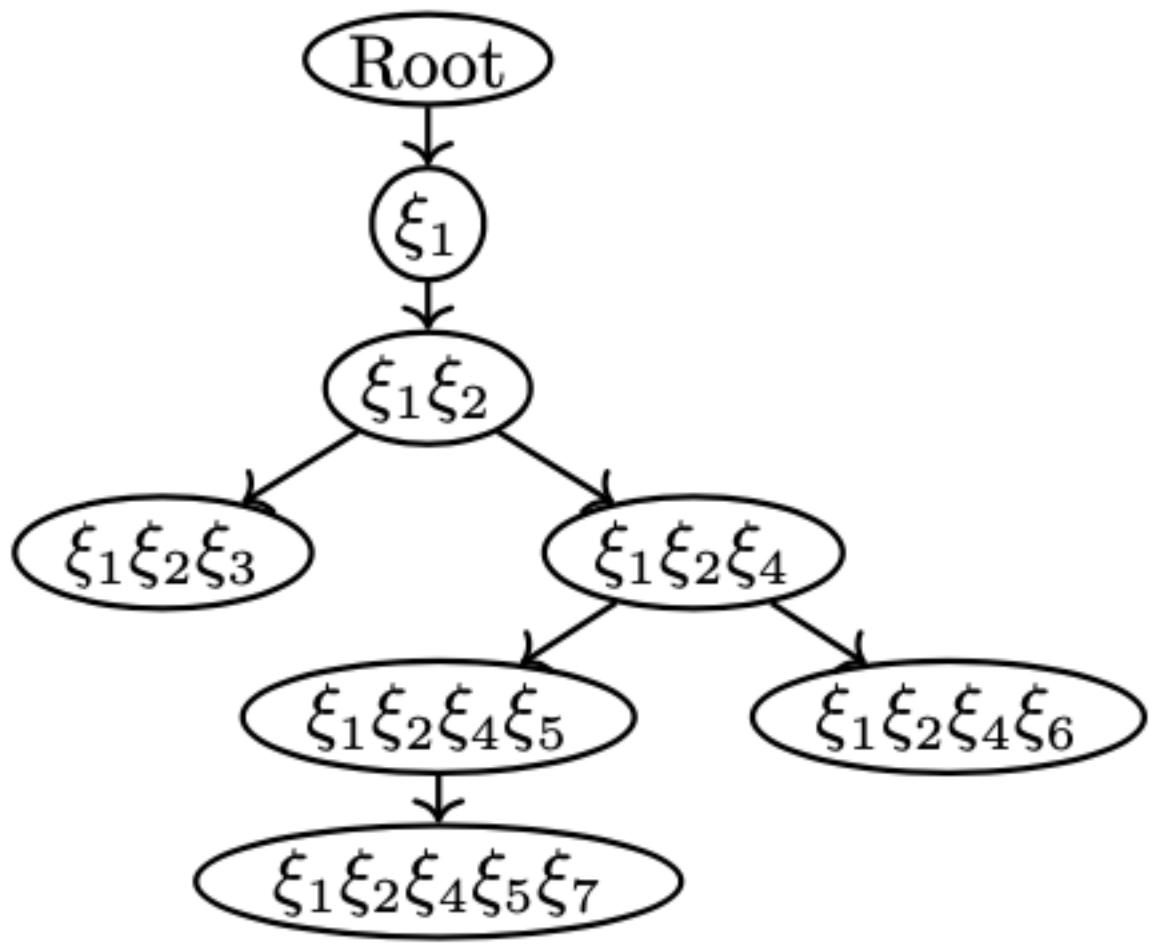
Key Result



Tree of Uncertainty Products

Key Result

$$\sum_{k\in[K]} q^{(k)}\xi_k + \sum_{n\in[N]} q_g^{(n)} \prod_{i\in S_n} \xi_i \geq q_0 \quad \forall \boldsymbol{\xi} \in \mathcal{U}$$



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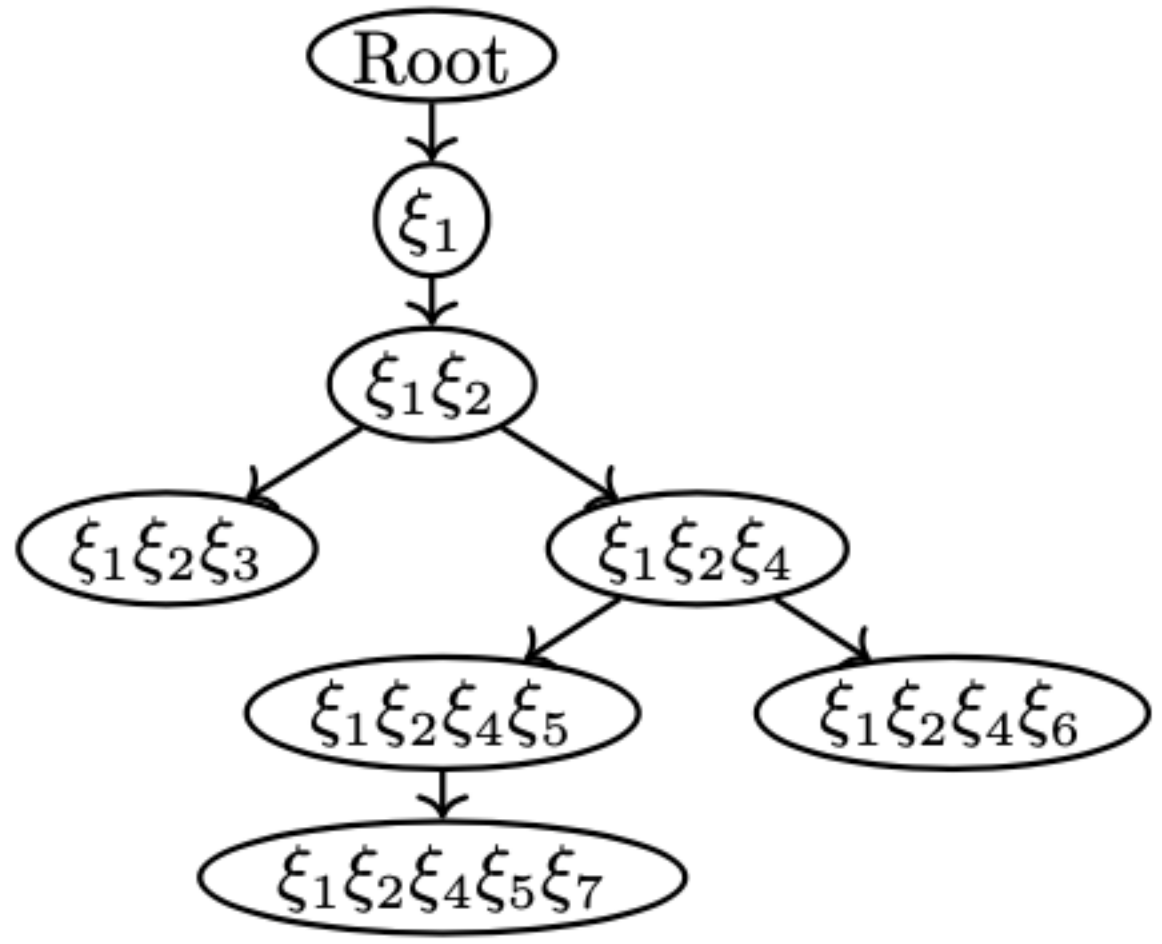
- Conservative Approximation

$$\sum_{k\in[K]} q^{(k)}\xi_k + \sum_{n\in[N]} q_g^{(n)} \eta_n \geq q_0 \quad \forall (\boldsymbol{\xi}, \boldsymbol{\eta}) \in \overline{\Xi}$$



$$\overline{\Xi} := \left\{ (\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbb{R}^{K+N} \;\middle|\; \begin{array}{ll} \boldsymbol{\xi} \in \Xi & \\ \eta_i = \xi_{k_i^*} & \forall i : \ell(i) = 0 \\ \eta_i \geq \overline{\eta}_{\ell(i)} \xi_{k_i^*} + \overline{\xi}_{k_i^*} \eta_{\ell(i)} - \overline{\eta}_{\ell(i)} \overline{\xi}_{k_i^*} & \forall i : \ell(i) \neq 0 \\ \eta_i \geq \underline{\eta}_{\ell(i)} \xi_{k_i^*} + \underline{\xi}_{k_i^*} \eta_{\ell(i)} - \underline{\eta}_{\ell(i)} \underline{\xi}_{k_i^*} & \forall i : \ell(i) \neq 0 \\ \eta_i \leq \overline{\eta}_{\ell(i)} \xi_{k_i^*} + \underline{\xi}_{k_i^*} \eta_{\ell(i)} - \overline{\eta}_{\ell(i)} \underline{\xi}_{k_i^*} & \forall i : \ell(i) \neq 0 \\ \eta_i \leq \underline{\eta}_{\ell(i)} \xi_{k_i^*} + \overline{\xi}_{k_i^*} \eta_{\ell(i)} - \underline{\eta}_{\ell(i)} \overline{\xi}_{k_i^*} & \forall i : \ell(i) \neq 0 \end{array} \right\}.$$

Tree of Uncertainty Products

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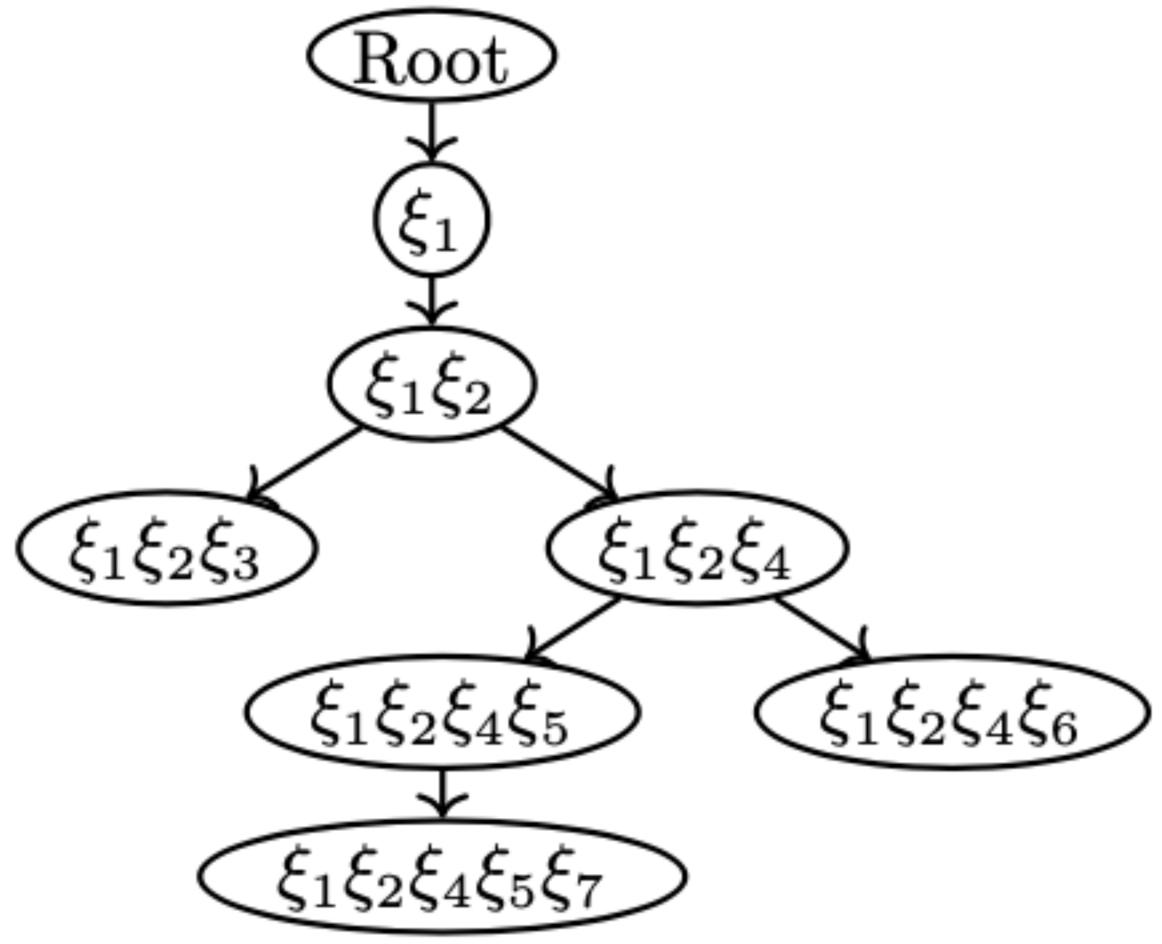
- Exact Reformulation
  - If $\mathscr{U}$ is a box and
  - Tree of Uncertainty Products has no overlap



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Tractable Approximations

Step 1. Employ **decision rules**, e.g., Linear decision rules

$$x_t^{(\tau)} = w_t^{(\tau)} + \sum_{t'=1}^{t-1} W_{t,t'}^{(\tau)} \theta_{t'} + \sum_{t'=1}^{t} \hat{W}_{t,t'}^{(\tau)} d_{t'}\,, C_t = v_t + \sum_{t'=1}^{t-1} V_{t,t'} \theta_{t'} + \sum_{t'=1}^{t-1} \hat{V}_{t,t'} d_{t'}.$$

Step 2. Substitute to the overall problem and identify all multilinear functions of uncertainties.

Step 3. Form a tree of uncertainty products and approximate with lifted uncertainty sets.

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Proposition

Under linear decision rules, the multistage problem is approximated as a static robust optimization problem with $\mathcal{O}(T^3)$ uncertain parameters and decision variables.

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Generalizable to *multilinear* decision rules!

Outline of Methods

- Robust Optimization (RO)
  - Uncertainties are described via polyhedral and box sets.
  - Decisions are made to minimize the worst-case cost.
  - Introduce the *tree of uncertainty products* and leverage McCormick relaxations to handle multilinear uncertainty.

$$\mathcal{U}_w(\mathbf{u}_T, \mathbf{x}_T) =$$
$$\{\mathbf{w}_t \mid \mathbf{w}_t = \rho_t(\mathbf{u}_t - x_t),\ \boldsymbol{\rho} \in \mathcal{U}_\rho\}$$
$$\mathbf{d} \in \mathcal{U}$$

- Distributionally Robust Optimization (DRO)
  - Uncertainties are described via unknown distributions, which are described via sets.
  - Decisions are made to minimize the worst-case expected cost.
  - Leverage the *mean-absolute deviation (MAD) based ambiguity sets*

$$F \in \mathcal{M}_+ \text{ s.t.}$$
$$\mathbf{P}_F\left(\xi_t \in [\underline{\xi}_t\ ,\ \overline{\xi}_t]\right) = 1$$
$$\mathbf{E}_F\left[|\xi_t - \widehat{\xi}_t|\right] \leq \lambda_{\xi_t}$$
$$\mathbf{E}_F[\xi_t] = \widehat{\xi}_t$$

- Numerical Experiments

Outline of DRO Approach

- Uncertainties are described via unknown distributions, which are described via ***ambiguity sets***.
- Decisions are made to minimize the worst-case expected cost.
- Overall formulation:

$$\min_{\mathbf{C}_B, C_1} \sup_{F_{d_1}} \mathbb{E}_{d_1}\left[\min_{\mathbf{x}_1} \sup_{F_{\mathbf{w}_1}} \mathbb{E}_{\mathbf{w}_1}\left[H_1(\cdot) + \min_{C_2} \sup_{F_{d_2}} \mathbb{E}_{d_2}\left[\min_{\mathbf{x}_2} \sup_{F_{\mathbf{w}_2}} \mathbb{E}_{\mathbf{w}_2}\left[H_2(\cdot) + \cdots + \min_{C_T} \sup_{F_{d_T}} \mathbb{E}_{d_T}\left[\min_{\mathbf{x}_T} \sup_{F_{\mathbf{w}_T}} \mathbb{E}_{\mathbf{w}_T}\left[H_T(\cdot)\right]\right] \cdots\right]\right]\right]\right]$$

Mean-MAD Ambiguity Sets

Definition

For the set of non-negative Borel measurable functions $\mathscr{M}_+(\mathbb{R}^{2T}), \lambda_{\theta_t}, \lambda_{d_t} \geq 0$, $0 \leq \underline{\theta}_t < \hat{\theta}_t < \overline{\theta}_t \leq 1$, and $0 \leq \underline{d}_t < \hat{d}_t < \overline{d}_t$, mean-MAD ambiguity set $\mathscr{F}$ is defined as

$$\mathcal{F} = \left\{ F \in \mathcal{M}_+(\mathbb{R}^{2T}) \;\middle|\; \begin{array}{ll} \mathbb{P}_F\left(\theta_t \in [\underline{\theta}_t, \overline{\theta}_t]\right) = 1,\ \mathbb{E}_F[\theta_t] = \widehat{\theta}_t,\ \mathbb{E}_F\left[|\theta_t - \widehat{\theta}_t|\right] \leq \lambda_{\theta_t} & \forall t \in [T] \\ \mathbb{P}_F\left(d_t \in [\underline{d}_t, \overline{d}_t]\right) = 1,\ \mathbb{E}_F[d_t] = \widehat{d}_t,\ \mathbb{E}_F\left[|d_t - \widehat{d}_t|\right] \leq \lambda_{d_t} & \forall t \in [T] \\ \{\theta_{[T]}, d_{[T]}\} \text{ are mutually independent} & \end{array} \right\}.$$

- $\underline{\theta}_t$, $\overline{\theta}_t$, $\underline{d}_t$, $\overline{d}_t$: lower and upper support of θ_t and d_t.
- $\hat{\theta}_t$, $\hat{d}_t$: expectation of θ_t and d_t.
- λ_{θ_t}, λ_{d_t} : mean-absolute deviation bound of θ_t and d_t.

All of them can be easily estimated from (small) data!

Mean-MAD Ambiguity Sets

$$\mathcal{F}_Y = \left\{F \in \mathcal{M}_+(\mathbb{R}) \;\middle|\; \mathbb{P}(\overline{Y} \in [\underline{y}, \overline{y}]) = 1,\; \mathbb{E}[\overline{Y}] = y_0,\; \mathbb{E}[|\overline{Y} - y_0|] \leq \overline{\lambda}\right\} \,,$$

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- The supremum of a convex function over this set is a 3 point distribution (Ben-Tal & Hochman 1972).

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- The supremum of a convex function over this set is a 3 point distribution (Ben-Tal & Hochman 1972).
- We extend existing results to the MAD set with inequality.
- This allows us to reformulate the DRO problem as a Stochastic Optimization problem.
  - We solve this problem using Sample Average Approximation

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Key Result

- The supremum of a convex function over this set is a 3 point distribution on
  - $\underline{y}, \overline{y}$ and y_0 with probabilities
  - $\dfrac{\overline{\lambda}}{2(y_0 - \underline{y})}, \dfrac{\overline{\lambda}}{2(\overline{y} - y_0)}$ and $1 - \dfrac{\overline{\lambda}}{2(y_0 - \underline{y})} - \dfrac{\overline{\lambda}}{2(\overline{y} - y_0)}$

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Multistage DRO → **Multistage SO**

Reformulation

Theorem

With the mean-MAD ambiguity set $\mathscr{F}$, the multistage DRO problem is reformulated as a stochastic optimization problem with *three-points discrete distributions* for each uncertain parameter.

Reformulation

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With the mean-MAD ambiguity set $\mathscr{F}$, the multistage DRO problem is reformulated as a stochastic optimization problem with *three-points discrete distributions* for each uncertain parameter.

$$\min_{\mathbf{C}_B, C_1} \sup_{F_{d_1}} \mathbb{E}_{d_1}\left[\min_{\mathbf{x}_1} \sup_{F_{\mathbf{w}_1}} \mathbb{E}_{\mathbf{w}_1}\left[H_1(\cdot) + \min_{C_2} \sup_{F_{d_2}} \mathbb{E}_{d_2}\left[\min_{\mathbf{x}_2} \sup_{F_{\mathbf{w}_2}} \mathbb{E}_{\mathbf{w}_2}\left[H_2(\cdot) + \cdots + \min_{C_T} \sup_{F_{d_T}} \mathbb{E}_{d_T}\left[\min_{\mathbf{x}_T} \sup_{F_{\mathbf{w}_T}} \mathbb{E}_{\mathbf{w}_T}\left[H_T(\cdot)\right]\right] \cdots\right]\right]\right]\right]$$

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Reformulation

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- Under mean and MAD constraints, the worst-case probability distribution is *always* fixed, supported over lower and upper bounds, and their means.
- **Insight**: There exists a class of stochastic optimization problems *whose solutions are distributionally robust*!

Outline of Methods

- Robust Optimization (RO)
  - Uncertainties are described via polyhedral and box sets.
  - Decisions are made to minimize the worst-case cost.
  - Introduce the *tree of uncertainty products* and leverage McCormick relaxations to handle multilinear uncertainty.

$$\mathcal{U}_w(\mathbf{u}_T, \mathbf{x}_T) = \{\mathbf{w}_t \mid \mathbf{w}_t = \rho_t(\mathbf{u}_t - x_t),\ \boldsymbol{\rho} \in \mathcal{U}_\rho\}$$

$$\mathbf{d} \in \mathcal{U}$$

- Distributionally Robust Optimization (DRO)
  - Uncertainties are described via unknown distributions, which are described via sets.
  - Decisions are made to minimize the worst-case expected cost.
  - Leverage the *mean-absolute deviation (MAD) based ambiguity sets*

$$F \in \mathcal{M}_+ \text{ s.t.}$$

$$\mathbf{P}_F\left(\xi_t \in [\underline{\xi}_t\ ,\ \overline{\xi}_t]\right) = 1$$

$$\mathbf{E}_F\left[|\xi_t - \widehat{\xi}_t|\right] \le \lambda_{\xi_t}$$

$$\mathbf{E}_F[\xi_t] = \widehat{\xi}_t$$

- Numerical Experiments

Case Study of Hernia Surgery

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  - Dates of office visit, surgery (if performed)
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- Our analysis estimates current backlog as 4 months of average (pre-pandemic) monthly demand.
- Four methods are implemented and compared:
  - RO: robust optimization-based method
  - DRO: distributionally robust optimization-based method
  - Det60: temporally increase capacity by at most 60% (for ~7 months)
  - Det100: temporally increase capacity by at most 100% (for ~5 months)

Performance Improvement

Departure Level	DRO		RO		Det60		Det100	
	Mean	CVaR90	Mean	CVaR90	Mean	CVaR90	Mean	CVaR90
More Departure	-3969 (10.0)	-2882 (6.31)	-3833 (6.25)	-2927 (7.97)	-2740 (-24.1)	-1871 (-31.0)	-3608 (0.0)	-2711 (0.0)

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Less Departure	-5078 (7.02)	-4284 (5.49)	-4906 (3.39)	-4306 (6.03)	-4078 (-14.1)	-3446 (-15.1)	-4745 (0.0)	-4061 (0.0)

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Structure of Expansion Policies

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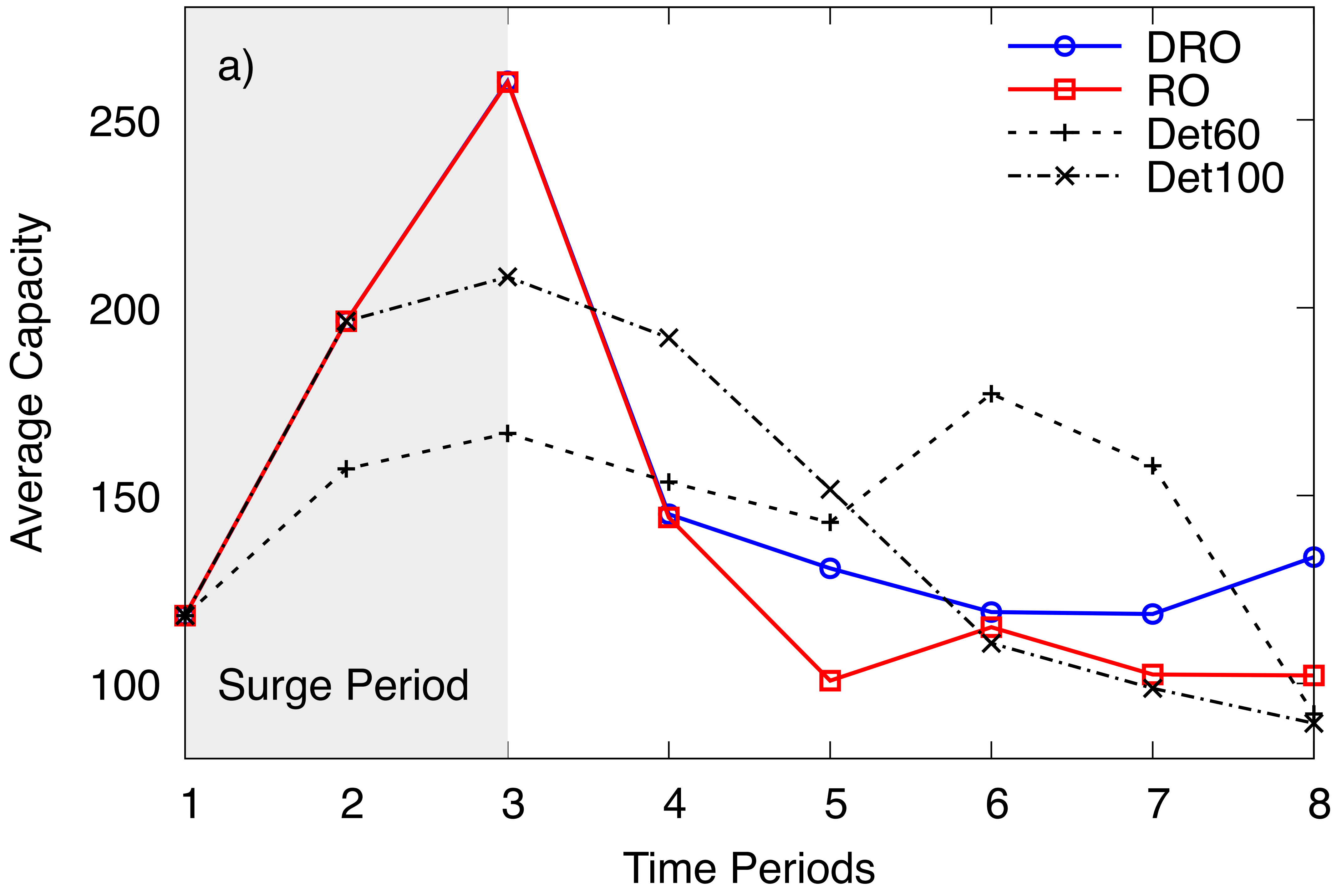
a) More Departure



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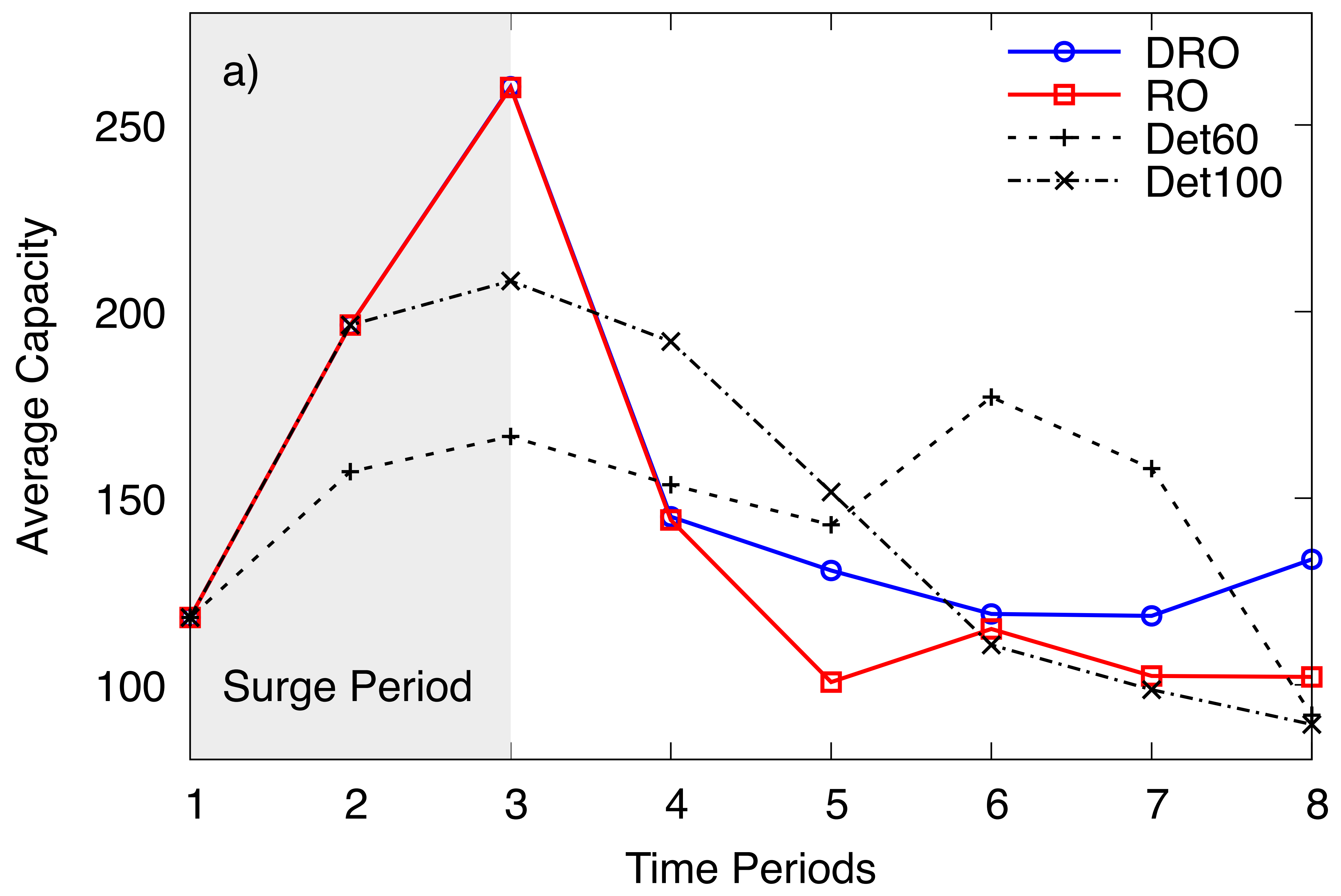

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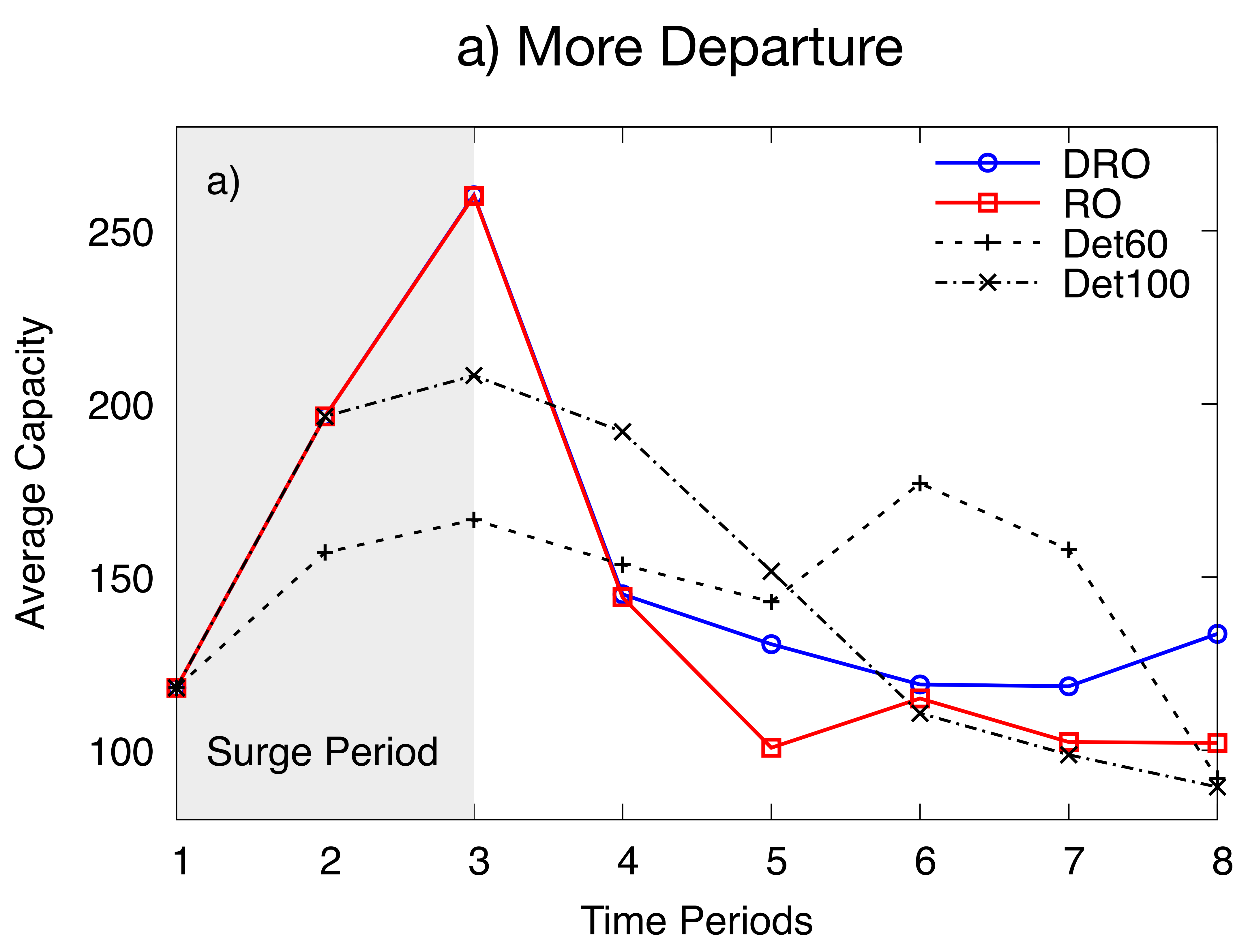

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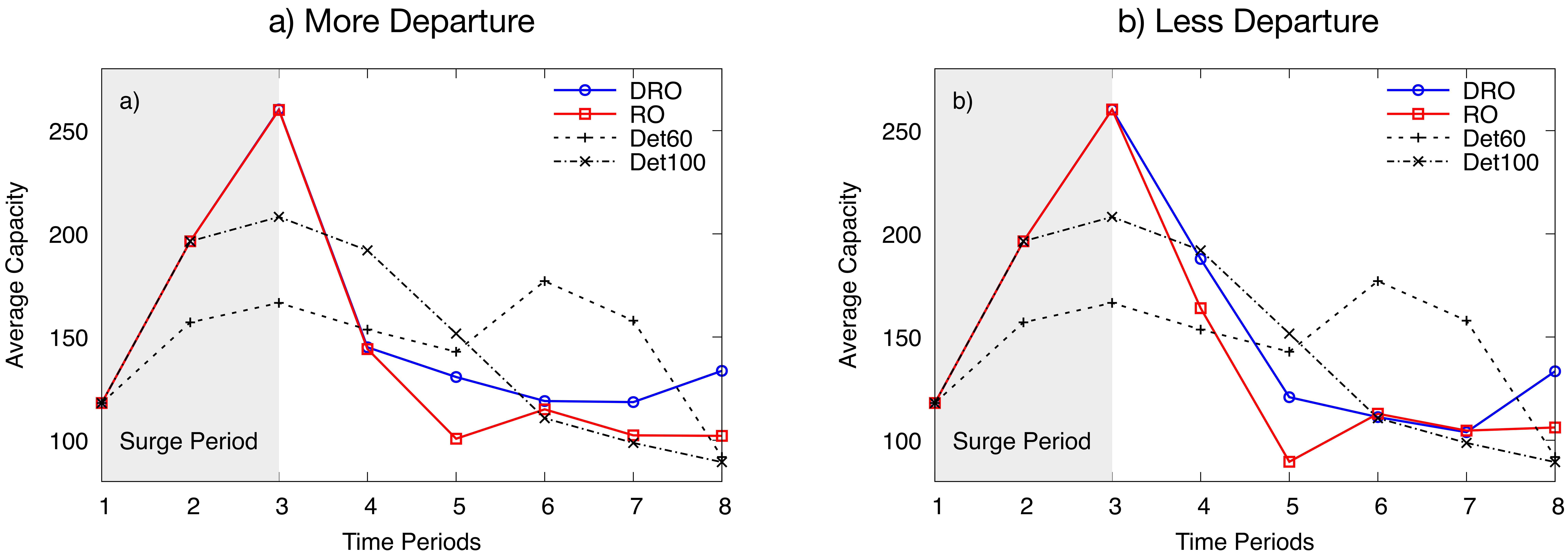

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Deferred and Departed Patients

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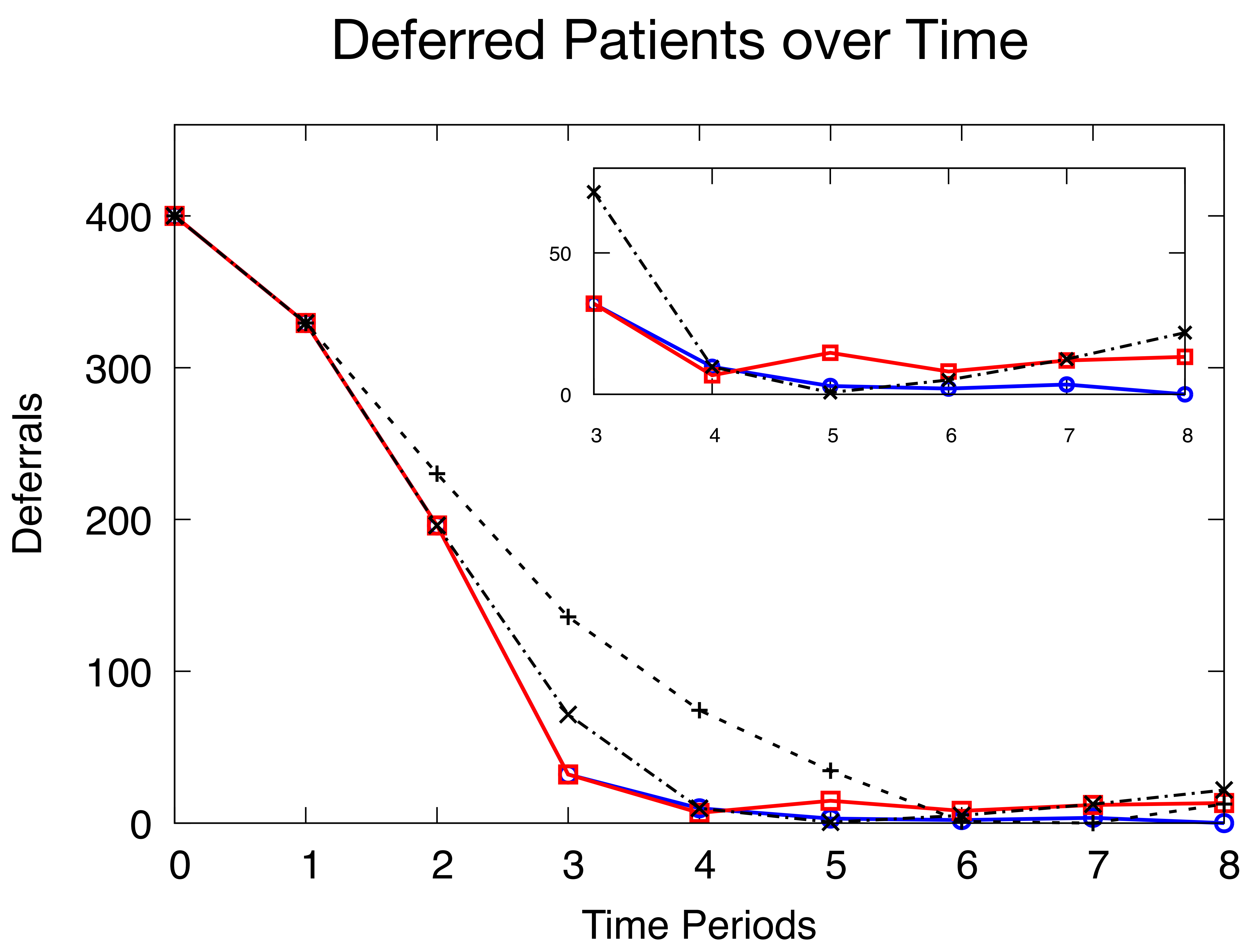

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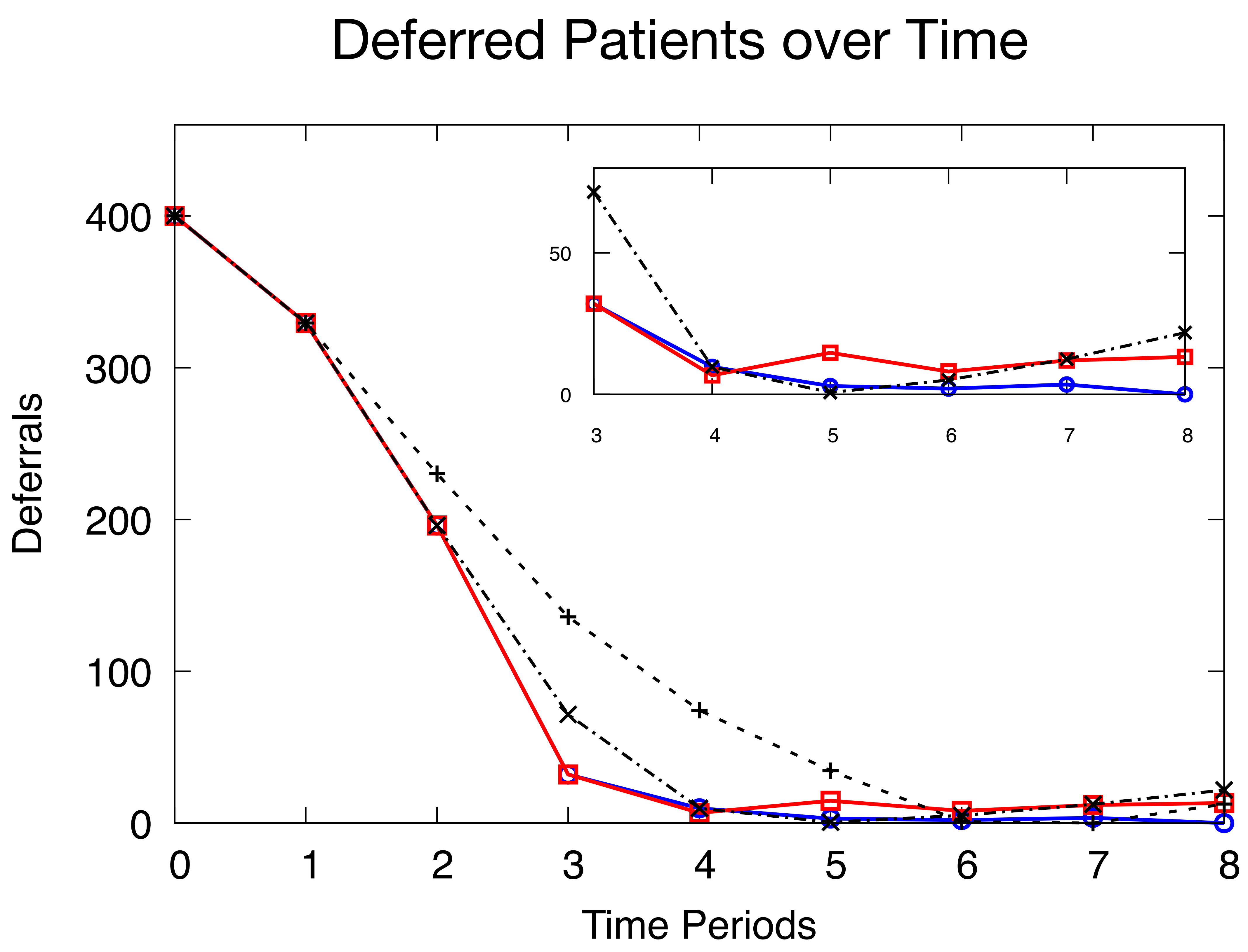

- Both RO and DRO policies achieve less numbers of deferrals and departures than deterministic policies.

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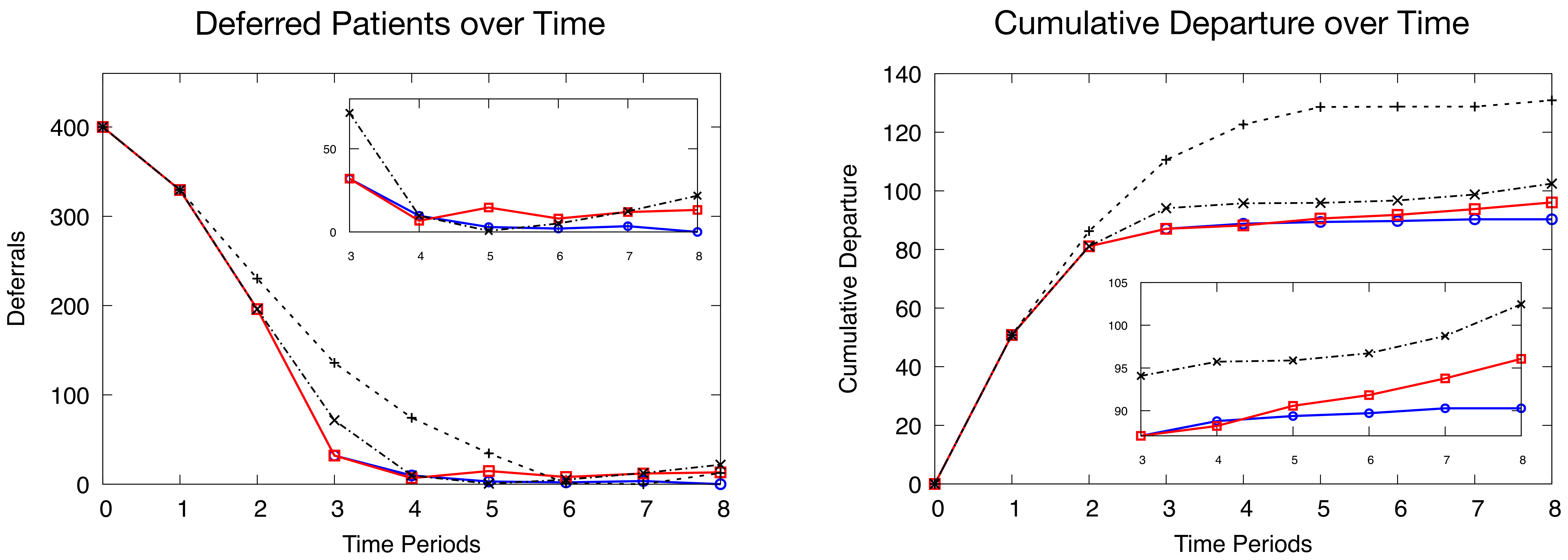


- Both RO and DRO policies achieve less numbers of deferrals and departures than deterministic policies.

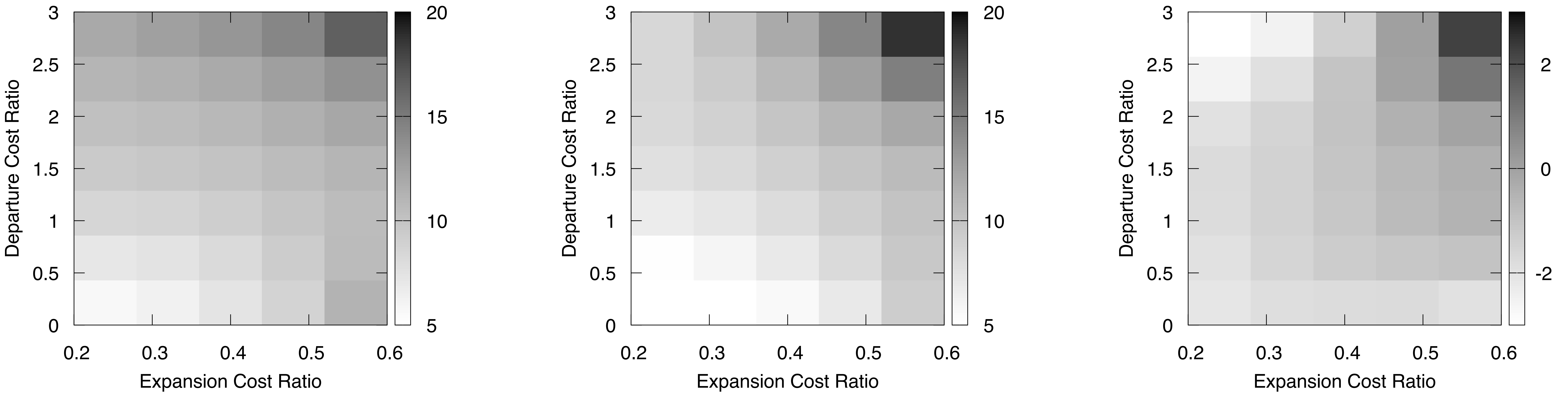
Comparison of Policies

	Lower demand (Mean 94)			Nominal demand (Mean 100)			Higher demand (Mean 106)		
	Static	Hybrid	Dynamic	Static	Hybrid	Dynamic	Static	Hybrid	Dynamic
RO	7.17%	8.91%	11.15%	5.61%	6.25%	9.90%	6.31%	7.89%	10.29%
DRO	2.61%	4.11%	4.36%	10.29%	10.00%	13.31%	11.12%	10.65%	13.10%
Det60	-23.8%	-24.2%	-24.9%	-25.6%	-24.1%	-23.3%	-27.4%	-26.6%	-26.2%

- Both RO and DRO policies improve over the deterministic policies
- RO is robust to higher and lower demand scenarios, but DRO is only guaranteed to protect against high demand settings.

Analysis of Outcomes

- Objective improvement (in percentage) over deterministic policies for different costs

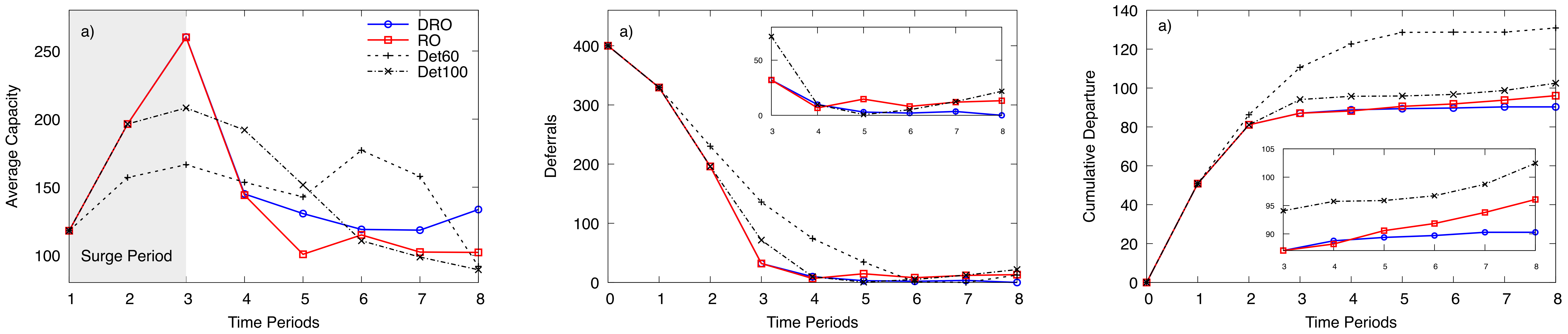


(a) Heatmap of DRO (b) Heatmap of RO (c) Heatmap of RO-DRO

- RO becomes more preferable than DRO when a decision-maker faces both higher expansion and departure costs.

Conclusions

- Dynamic expansion of surgical capacity is necessary to manage a large number of deferred surgeries.
- We develop two optimization methods, based on RO and DRO.
- We introduce the notion of tree of uncertainty products to make RO models tractable.
- Proposed methods significantly improve objectives (5~10%) over deterministic policies in the hernia case study.



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