Optimization under Decision Dependent Uncertainty

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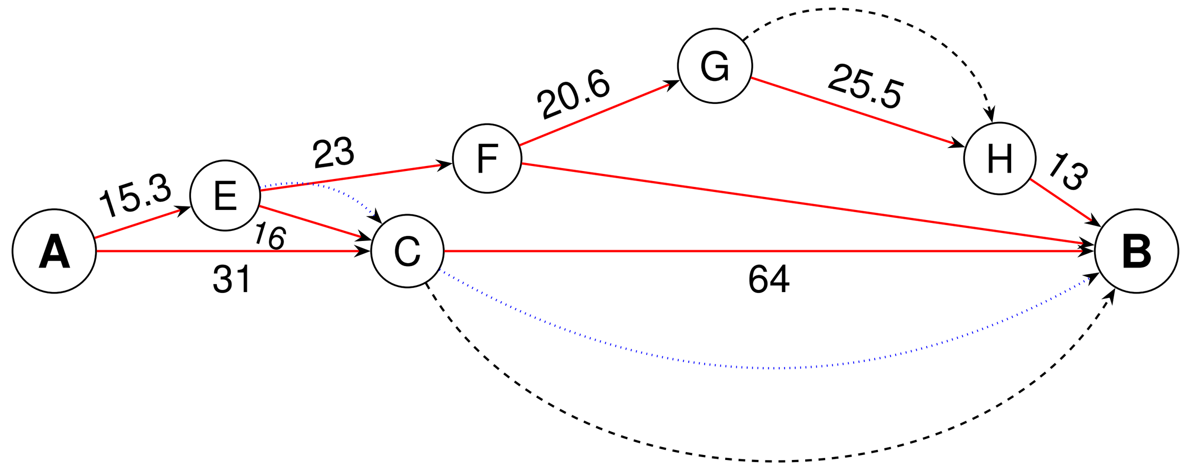
Abstract

In many applications, uncertainties are affected by decisions. These dependencies are not modeled by current frameworks.

- We show RO problems with decision-dependent uncertainties (DDU) are NP-complete.
- We introduce a new class of uncertainty sets with decision-dependent sizes and with good reformulations.
- Application: Shortest Path with uncertain arc lengths.

Introduction

In the graph the length of any arc e is uncertain $d_e = \bar{d}_e(1+\xi_e)$. The uncertainty ξ_e is in the set $\mathcal{U}(\mathbf{x}) = \{\boldsymbol{\xi} \mid 0 \le \xi_e \le 1 - 0.8x_e\}$ where the binary decision x_e determines whether to reduce the maximum possible uncertainty. The objective is to find the shortest path from A to B.



Shortest Path	Path	Nominal	Worstcase
Nominal	A–C–B	95	127
Robust	A–E–F–G–H–B	97.4	110.15
DDU	A–E–C–B	95.3	108.1

Decision-dependent sets can mitigate the conservatism of worst case scenarios.

Complexity

RO-DDU :
$$\min_{\mathbf{x},\mathbf{y}} \ \mathbf{c}^\top\mathbf{x} + \mathbf{f}^\top\mathbf{y}$$
$$\text{s.t. } \mathbf{a}_i^\top\mathbf{x} + \boldsymbol{\xi}_i^\top\mathbf{y} \le b_i \quad \forall \boldsymbol{\xi}_i \in \mathcal{U}_i^P(\mathbf{x}) \subseteq \mathbb{R}^n \quad \forall i$$

Uncertainty set : $\mathcal{U}^P(\mathbf{x}) = \{\boldsymbol{\xi} \mid \mathbf{D}\boldsymbol{\xi} \le \mathbf{d} + \boldsymbol{\Delta}\mathbf{x}\}$

Theorem 1. *The robust linear problem (RO-DDU) with uncertainty set $\mathcal{U}^P$ is NP-complete.*

Structured Uncertainty Sets

The following set captures common models of uncertainty and incorporates decision dependence.

$\overline{\mathbf{\Pi}}$-Uncertainty:

$$\mathcal{U}^{\overline{\mathbf{\Pi}}}(\mathbf{x}) = \{\boldsymbol{\xi} \mid \mathbf{D}\boldsymbol{\xi} \le \mathbf{d}, \ \ \boldsymbol{\xi} \le \mathbf{v} + \mathbf{W}(\mathbf{e} - \mathbf{x}), \ \ \boldsymbol{\xi} \ge \mathbf{0}\}$$

Note: $\mathcal{U}^{\overline{\mathbf{\Pi}}}(\mathbf{x})$ is an intersection of a polyhedron and a box. The size of the box changes with the decision $\mathbf{x}$.

Goal: We reformulate the constraint
$$\mathbf{y}^\top\boldsymbol{\xi} \le b \ \ \forall \boldsymbol{\xi} \in \mathcal{U}^{\overline{\mathbf{\Pi}}}(\mathbf{x}) \qquad \text{(LC)}$$

Worst case scenario for constraint (LC)

$$h(\mathbf{x},\mathbf{y}) = \max_{\boldsymbol{\xi}} \ \mathbf{y}^\top\boldsymbol{\xi} \quad \text{s.t. } \mathbf{D}\boldsymbol{\xi} \le \mathbf{d}, \ \ \boldsymbol{\xi} \le \mathbf{v} + \mathbf{W}(\mathbf{e}-\mathbf{x}), \ \ \boldsymbol{\xi} \ge \mathbf{0}$$

$$\bar{h}(\mathbf{x},\mathbf{y}) = \max_{\boldsymbol{\xi},\boldsymbol{\zeta}} \ (\mathbf{y} - \overline{\mathbf{\Pi}}\mathbf{x})^\top\boldsymbol{\xi} + \mathbf{y}^\top\boldsymbol{\zeta} \quad \text{s.t. } \mathbf{D}\boldsymbol{\xi} + \mathbf{D}\boldsymbol{\zeta} \le \mathbf{d}, \ \ \boldsymbol{\xi} \le \mathbf{W}\mathbf{e}, \ \ \boldsymbol{\zeta} \le \mathbf{v}, \ \ \boldsymbol{\xi},\boldsymbol{\zeta} \ge \mathbf{0}$$

Theorem 2. *For a binary $\mathbf{x}$, if the set $\mathcal{U}^{\overline{\mathbf{\Pi}}}(\mathbf{x})$ is nonempty and $\mathbf{v}, \mathbf{W} \ge 0$, then for all $\mathbf{y}$:*

$$h(\mathbf{x},\mathbf{y}) = \bar{h}(\mathbf{x},\mathbf{y}).$$

Using Theorem 2, the constraint (LC) can be expressed as

$$\mathbf{t}^\top\mathbf{d} + \mathbf{r}^\top\mathbf{W}\mathbf{e} + \mathbf{s}^\top\mathbf{v} \le b$$
$$\mathbf{s}^\top + \mathbf{t}^\top\mathbf{D} \ge \mathbf{y}^\top$$
$$\mathbf{r}^\top + \mathbf{t}^\top\mathbf{D} \ge \mathbf{y}^\top - \mathbf{x}^\top\overline{\mathbf{\Pi}}$$
$$\mathbf{r}, \mathbf{s}, \mathbf{t} \ge \mathbf{0}$$

→ This has fewer constraints than the Big-M reformulation.

Application: Shortest Path

$$\min_{\mathbf{x},\mathbf{y}} \max_{\boldsymbol{\xi} \in \mathcal{U}^{SP}(\mathbf{x})} \sum_{(i,j)\in\mathcal{A}} c_{ij}x_{ij} + \sum_{(i,j)\in\mathcal{A}} d_{ij}(\boldsymbol{\xi})y_{ij} \qquad \text{(SP)}$$
$$\text{s.t. } \mathbf{x} \in X \subseteq \{0,1\}^{|\mathcal{A}|}, \ \mathbf{y} \in Y$$

The length of the arc (i,j) is uncertain $d_{ij}(\boldsymbol{\xi}) = \bar{d}_{ij}(1+\xi_{ij})$.
x_{ij} : Whether to reduce the uncertainty in the arc (i,j).
y_{ij} : Whether arc (i,j) is in the shortest path.

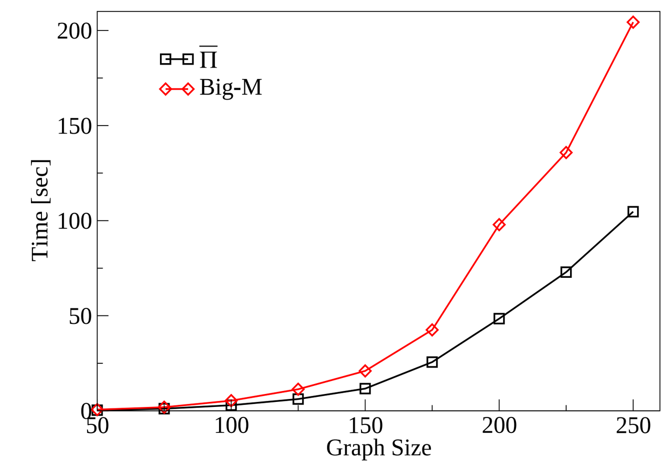
Uncertainty Set:

$$\mathcal{U}^{SP}(\mathbf{x}) = \left\{ \boldsymbol{\xi} \mid \sum_{(i,j)\in\mathcal{A}} \xi_{ij} \le \Gamma, \ \xi_{ij} \le \ 1 - \gamma_{ij}x_{ij}, \ \xi_{ij} \ge 0 \ \ \forall (i,j) \right\}$$

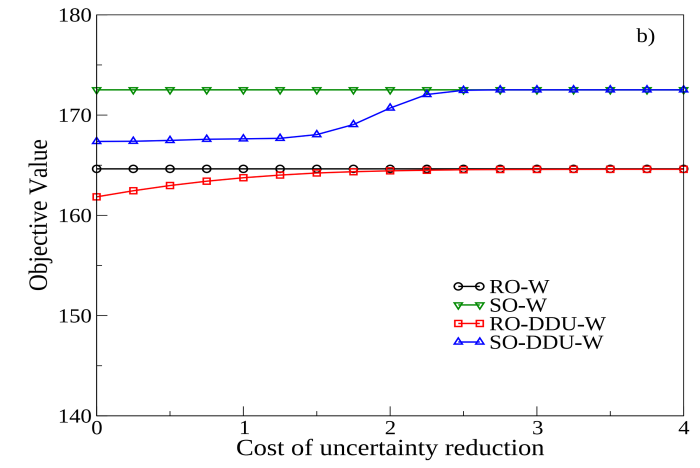
Numerical Experiment

Numerical Setup: 100 randomly generated graphs, $c = 1.0, \ \gamma = 0.2, \Gamma = 2$

Performance



Cost of Reduction



→ The $\overline{\mathbf{\Pi}}$ reformulation scales better than Big-M.
→ DDU objective value improves over RO and SO for $c < 2$.

Conclusion

We present an alternative model to reduce the conservatism of robust optimization problems which improves the worst case scenario. We prove the difficulty of decision-dependent problems and introduce a new uncertainty set to capture the decision-dependent behavior while still ensuring good reformulation capabilities.