# Tractable Optimization for Multilinear Uncertainty

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# Results are joint work with...

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# Motivation

- Presence of Products of Uncertain terms.

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- **Compounding**: The initial quantity increases by a percentage in every time period. Examples: interest, economic and population growth etc.
- **Depreciation and Shrinkage**: The initial quantity decreases by a fixed fraction over time. Examples: Financial depreciation, damage, loss due to theft

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- Objective functions and constraints involve multilinear uncertainty terms. General representation

$$\mathbf{p}^\top \boldsymbol{\xi} + \sum_{n=1}^{N} q_n g_n(\boldsymbol{\xi}) \geq q_0 \quad \forall \boldsymbol{\xi} \in \Xi \text{ where } g_n(\xi) = \prod_{i \in \mathcal{J}_n} \xi_i$$

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- Example

$$\mathbf{p}^\top \xi + q_1 \xi_1 \xi_2 + q_2 \xi_2 \xi_3 + q_3 \xi_1 \xi_2 \xi_3 \geq q_0 \ \ \forall \boldsymbol{\xi} \in \Xi$$

  Here, $\mathcal{J}_1 = \{1, 2\}, \ \mathcal{J}_2 = \{2, 3\}, \ \mathcal{J}_3 = \{1, 2, 3\}$
- Nonlinear and non-convex function of the uncertainty.

# Existing Results

Various works have looked at reformulation constraints with different types of uncertainty dependence

- Bilinear Uncertainty: Peng et al. (2017), Zhen et al. (2021)
- Non-convex quadratic Uncertainty: Xu and Hanasusanto (2023)
- Multilinear uncertainty: Georghiou et al. (2015)
- Convex relaxations of multilinear terms: Ryoo and Sahinidis (2001), Luedtke et al. (2012)

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- **Observation**: Constraints with Bilinear Uncertainty can be tractably reformulated for box uncertainty sets
- **Idea**: Identify settings in which the collection of multilinear terms in a constraint can be expressed as a sequence of bilinear terms
- **Example**: Consider the constraint

$$x_1\xi_1\xi_2 + x_2\xi_1\xi_2\xi_3 \geq 1 \ \forall \boldsymbol{\xi} \in \Xi$$

Then

$$x_1\eta_1 + x_2\eta_2 \geq 1 \ \forall(\eta_1, \eta_2, \boldsymbol{\xi}) \in \Xi' := \{(\boldsymbol{\xi},\ \xi_1\xi_2,\ \xi_1\xi_2\xi_3) : \boldsymbol{\xi} \in \Xi\}$$

where $\eta_1 = \xi_1\xi_2$ and $\eta_2 = \eta_1\xi_3$.

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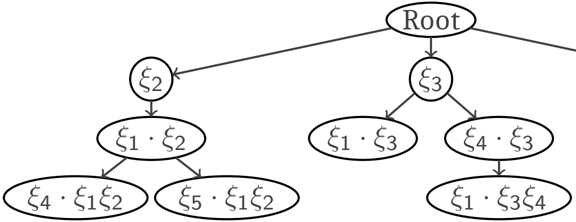
- At each node, we add a new uncertain expression to the linear or bilinear expression present in the parent node.



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$$\Xi := \left\{ (\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbb{R}^{K+N} \;\middle|\; \begin{array}{ll} \boldsymbol{\xi} \in \Xi,\ \eta_i = \xi_{k_i^*} & \forall i : \ell(i) = 0 \\ \eta_i \geq \overline{\eta}_{\ell(i)} \xi_{k_i^*} + \overline{\xi}_{k_i^*} \eta_{\ell(i)} - \overline{\eta}_{\ell(i)} \overline{\xi}_{k_i^*} & \forall i : \ell(i) \neq 0 \\ \eta_i \geq \underline{\eta}_{\ell(i)} \xi_{k_i^*} + \underline{\xi}_{k_i^*} \eta_{\ell(i)} - \underline{\eta}_{\ell(i)} \underline{\xi}_{k_i^*} & \forall i : \ell(i) \neq 0 \\ \eta_i \leq \overline{\eta}_{\ell(i)} \xi_{k_i^*} + \underline{\xi}_{k_i^*} \eta_{\ell(i)} - \overline{\eta}_{\ell(i)} \underline{\xi}_{k_i^*} & \forall i : \ell(i) \neq 0 \\ \eta_i \leq \underline{\eta}_{\ell(i)} \xi_{k_i^*} + \overline{\xi}_{k_i^*} \eta_{\ell(i)} - \underline{\eta}_{\ell(i)} \overline{\xi}_{k_i^*} & \forall i : \ell(i) \neq 0 \end{array} \right\}.$$

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# Results

$$\mathbf{p}^\top \boldsymbol{\xi} + \sum_{n=1}^{N} q_n g_n(\boldsymbol{\xi}) \geq q_0 \quad \forall \boldsymbol{\xi} \in \Xi \qquad (1)$$

$$\mathbf{p}^\top \boldsymbol{\xi} + \sum_{n=1}^{N} q_n \eta_n \geq q_0 \quad \forall (\boldsymbol{\eta}, \boldsymbol{\xi}) \in \overline{\Xi} \qquad (2)$$

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**Theorem**

*The constraint (2) is a conservative reformulation of the constraint (1).*

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*If $\Xi = \bigtimes_{n=1}^{N} [0, \bar{\xi}_n]$ and the tree of uncertain products is such that every uncertain element is only added to a single node then constraint (1) is equivalent to (2).*

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- This uses the fact that for bilinear terms, the nonlinear set $\{(\xi_1, \xi_2, \xi_1, \xi_2) : \boldsymbol{\xi} \in \Xi\}$ is a subset of the lifted set constructed using the McCormick envelopes.

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- We then show that for every multilinear expression, the product term at each node in the path to the multilinear term is included in the lifted set using induction and the previous bilinear result.

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- Show that in the original constraint we can replace the uncertainty set by its convex hull. This is done using the Caratheodory Theorem.
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- Show that in the original constraint we can replace the uncertainty set by its convex hull. This is done using the Caratheodory Theorem.
- Show that $Conv(\{(\xi, g_n(\xi)) : \xi \in \Xi\}) \subseteq \overline{\overline{\Xi}}$. This is true due to Theorem 1.
- Show that $\overline{\overline{\Xi}} \subseteq Conv(\{(\xi, g_n(\xi)) : \xi \in \Xi\})$.
  - This is done by showing that it is sufficient to show the result for the unit cube.
  - Show that for the unit cube, every extreme point of $\overline{\overline{\Xi}}$ lies in $Conv(\{(\xi, g_n(\xi)) : \xi \in \Xi\})$.

# Decision Rule Usage

So far $\mathbf{p}$ and $q_n$ have been fixed and independent of $\xi$.

$$\mathbf{p}^\top \xi + \sum_{n=1}^{N} q_n g_n(\xi) \geq q_0 \quad \forall \xi \in \Xi$$

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So far $\mathbf{p}$ and $q_n$ have been fixed and independent of $\boldsymbol{\xi}$.

$$\mathbf{p}^\top \boldsymbol{\xi} + \sum_{n=1}^{N} q_n g_n(\boldsymbol{\xi}) \geq q_0 \quad \forall \boldsymbol{\xi} \in \Xi$$

If $\mathbf{p}$ and $q_n$ are linear or multilinear functions of the uncertainty

$$\mathbf{p} = \mathbf{p_0} + \mathbf{P}\boldsymbol{\xi}, \; q_n = q_n^0 + \mathbf{Q}_n^\top \boldsymbol{\xi}$$

$$\mathbf{p} = \mathbf{p_0} + \mathbf{P} \prod_{j \in \mathcal{J}} \xi_j, \; q_n = q_n^0 + Q_n \prod_{j \in \mathcal{J}_n} \xi_j$$

The constraint remains multilinear and can be reformulated as before.

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- These results are used to reformulate a robust dynamic capacity management problem which captures queuing behaviour and balking.

*Han, E., Sharma, K., Singh, K., and Nohadani, O. Dynamic Capacity Management for Deferred Surgeries*

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Thank you for your attention!

# Tightening for Non-Rectangular Settings

If $\Xi$ is non-negative, compact and convex and lies in $\bigtimes_{k\in[K]}[\underline{\xi}_k, \overline{\xi}_k]$ Then, we can obtain a tighter representation for the lifted uncertainty set using results by Astreicher et al. (2021)

$$\begin{cases} \eta_i \leq \overline{\eta}_i^* \\ \overline{\eta}_i^*(\eta_i - \underline{\xi}_i \underline{\eta}_{\ell(i)})^2 \leq \left(\overline{\eta}_i^*(\xi_i - \underline{\xi}_i) + \underline{\xi}_i(\eta_i - \underline{\eta}_{\ell(i)}\xi_i)\right)\left(\overline{\eta}_i^*(\eta_{\ell(i)} - \underline{\eta}_{\ell(i)}) + \underline{\eta}_{\ell(i)}(\eta_i - \underline{\xi}_i\eta_{\ell(i)})\right) \end{cases} \forall$$