Data-driven Distributionally Robust Optimization over Time

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Joint work with Kevin-Martin Aigner, Kristin Braun, Sebastian Pokutta, Frauke Liers, Andreas Barman, Oscar Schneider, and Sebastian Tschupik

Motivation

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- DRO problems are difficult to solve
  - Dualization increases the size of the problem
  - It also removes any special structure

Existing Work

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DRO over Time

- Uses recomputation of exact solution with updated ambiguity set (Bayraksan and Love 2015, Esfahani and Kuhn 2018, Kirschner et. al. 2021 …)

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Faster Solutions Methods

- Convert DRO problem into a regularisation problem (Namkoong and Duchi 2016, Chen et al 2017, Levy et al 2020 …)

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Contributions

- We provide an Online Algorithm for DRO that simultaneously learns the uncertainty set and converges to the optimal solution
- We prove the consistency of our algorithm and also bound its regret over time
- We illustrate the performance of our method through numerical experiments on benchmark libraries.

Model

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Learning

- Observe samples from a distribution p^* (**finite dimension**)
- Use samples to construct ambiguity set $\mathscr{P}_0$ containing p^* with high probability

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- Leverage ambiguity set $\mathscr{P}_t$ to make robust decisions x_t

Learning

- Use new observations to update the set $\mathscr{P}_{t+1}$ and repeat

Ambiguity Sets

- Confidence Interval Ambiguity Set

$$\mathcal{P}_t = \left\{ p \in \mathcal{P}_0 \mid |p - \hat{p}_t| \leq \frac{z_{\frac{\delta_t}{2}}}{\sqrt{t}} \right\}$$

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- ℓ_2−norm Ambiguity Set

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- Kernel based Ambiguity Set

$$\mathcal{P}_t = \left\{ p \in \mathcal{P}_0 \mid \|p - \hat{p}_t\|_M \leq \frac{\sqrt{C}}{\sqrt{t}}(2 + \sqrt{2\log(1/\delta_t)}) \right\}$$

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- The sets include the true distribution with high probability
- They shrink fast enough to compensate for the increasing probability requirements

Standard Reformulation of DRO

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- Dual Reformulation for Interval Ambiguity Sets

$$\begin{aligned}
\min_{x,z,\alpha,\beta} \quad & z - \langle l_t, \alpha \rangle + \langle u_t, \beta \rangle \\
\text{s.t.} \quad & z - \alpha_k + \beta_k \geq f(x, s_k) \quad \forall k = 1, \dots, |\mathcal{S}|, \\
& \alpha, \beta \geq 0, \\
& x \in \mathcal{X}, \; z \in \mathbb{R}, \; \alpha, \beta \in \mathbb{R}^{|\mathcal{S}|}.
\end{aligned}$$

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Algorithm 1 DRO over Time

Input: functions $f(\cdot, s)$ for $s \in \mathcal{S}$, feasible set $\mathcal{X}$, initial amb

Output: sequence of DRO solutions $x_1, \dots, x_T$

for $t = 1$ **to** T **do**

  $x_t \leftarrow$ solve Problem (1) or (2) for $\mathcal{P}_{t-1}$

  $\mathcal{P}_t \leftarrow$ observe data and update set parameters such as ambiguity set.

end for

Online Optimization

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- Step by decision maker x $$x_t = \arg\min_{x \in \mathcal{X}} \mathbb{E}_{s \sim p_t} [f(x, s)]$$

Online Optimization

Algorithm 2 DRO over Time with Online Projected Gradient Descent

1: **Input:** functions $f(\cdot, s)$ for $s \in \mathcal{S}$, feasible set $\mathcal{X}$, initial ambiguity set $\mathcal{P}_0$

2: **Output:** $x_1, \ldots, x_T$

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Proof technique:

- True distribution inside ambiguity set
- The ambiguity set converges to the true distribution

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Regret Bound on Solutions

With probability at least $1 - \delta$ we have

$$\frac{1}{T}\sum_{t=1}^{T}\left(\max_{p\in\mathcal{P}_t}\mathbb{E}_{s\sim p}\left[f(x_t,s)\right]-\min_{x\in\mathcal{X}}\max_{p\in\mathcal{P}_t}\mathbb{E}_{s\sim p}\left[f(x,s)\right]\right)\leq G\sqrt{\frac{|\mathcal{S}|h(T)}{2T}}+\frac{2G}{T},$$

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- Linear Dependence on Scenarios

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- Bound the regret term by the linear drop in the function value

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- Bound the cumulative linear drop in function value by a bound on the gradient and the cumulative length of the steps taken

$$\sum_{t=1}^{T}\langle\eta\nabla g_t(p_t),p_t-u_t\rangle\leq\sum_{t=1}^{T}\frac{\eta^2}{2}\|\nabla g_t(p_t)\|^2+\sum_{t=1}^{T}\left(\frac{1}{2}\|p_t-u_t\|^2-\frac{1}{2}\|p_{t+1}-u_t\|^2\right)$$

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- Bound the cumulative length of the steps on the basis of the uncertainty set size.

$$\sum_{t=1}^{T}\left(\frac{1}{2}\|p_t-u_t\|^2-\frac{1}{2}\|p_{t+1}-u_t\|^2\right)\leq\sum_{t=1}^{T}\frac{1}{2}\|p_t-u_t\|^2$$

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$$\frac{1}{2}\sum_{t=1}^{T}\|p_t - q_t\|^2 \leq h(T)$$

$$p_t \in \mathcal{P}_{t-1}, q_t \in \mathcal{P}_t$$

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Confidence Intervals $h(T) = 8|\mathcal{S}|\log(\pi T)(1+\log T)$

ℓ_2-norm Sets $h(T) = 8|\mathcal{S}|\log\frac{\pi T}{\sqrt{3\delta}}(1+\log T)$

Kernel based Sets $h(T) = \frac{32C}{\lambda^2} + \frac{32C}{\lambda^2}\log\frac{\pi T}{\sqrt{6\delta}}(1+\log T)$

Numerical Experiments

Benchmark Libraries

- MILPs and MIQPs from the MIPLIB set of benchmark Instances
- Comparisons against other methods

Distributionally Robust Network Design

- Network design with uncertain demands. Instances by Altin et. al. (2007)

Optimal Route Choice

- *ChicagoSketch* model from Transportation Networks library
- Illustration of impact on solutions

Benchmark Libraries

- Takes MILP and MIQP instances from MIPLIB library
- Instances are of the form

$$f(x, s) = x^\top Q x + (c + s)^\top x + d$$

- **Objective** uncertainty in the instances through scenarios $s \in \mathcal{S}$ with $|\mathcal{S}| = (2, 10, 15)$

Benchmark Instances: Different Ambiguity Sets

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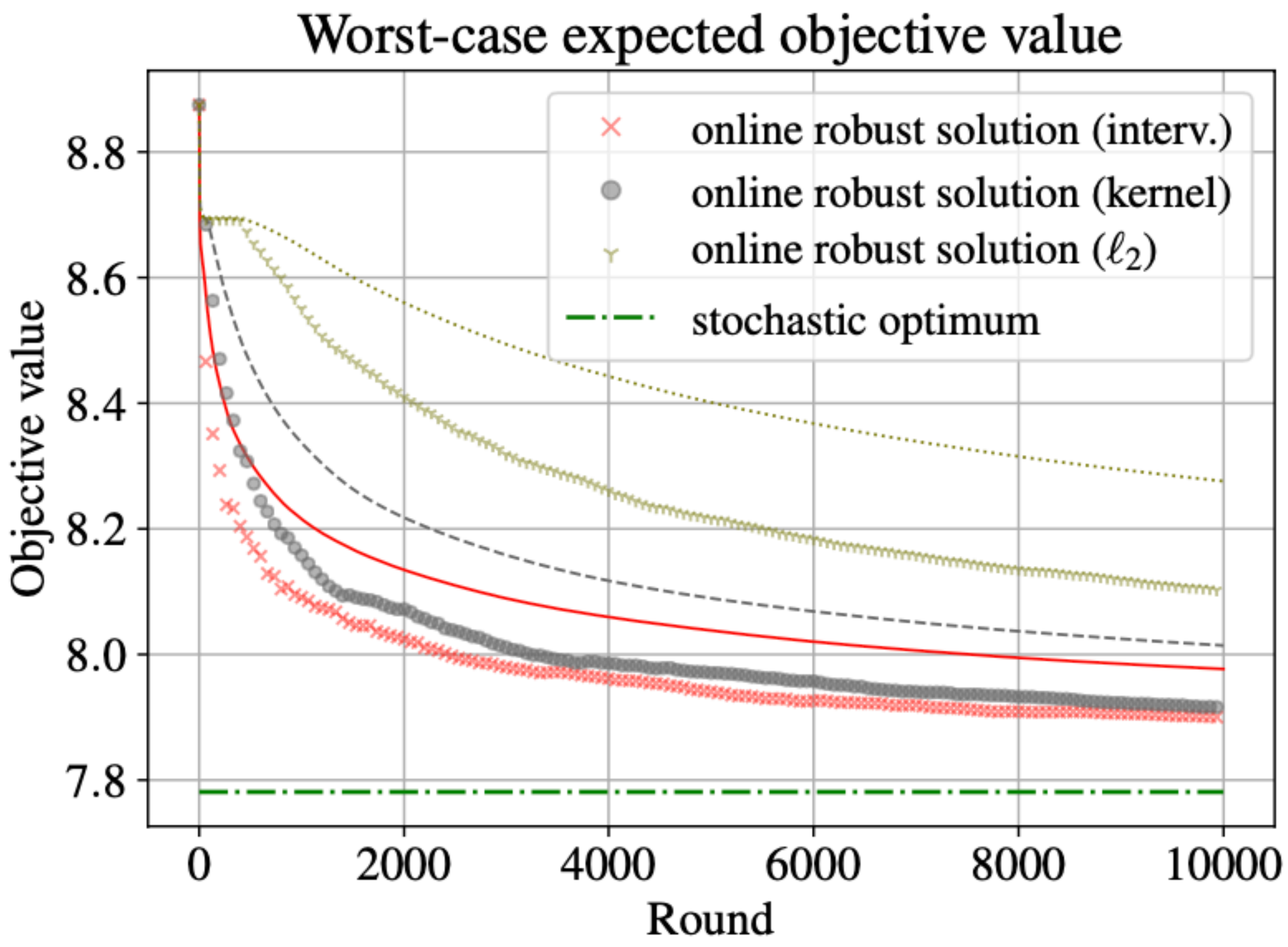

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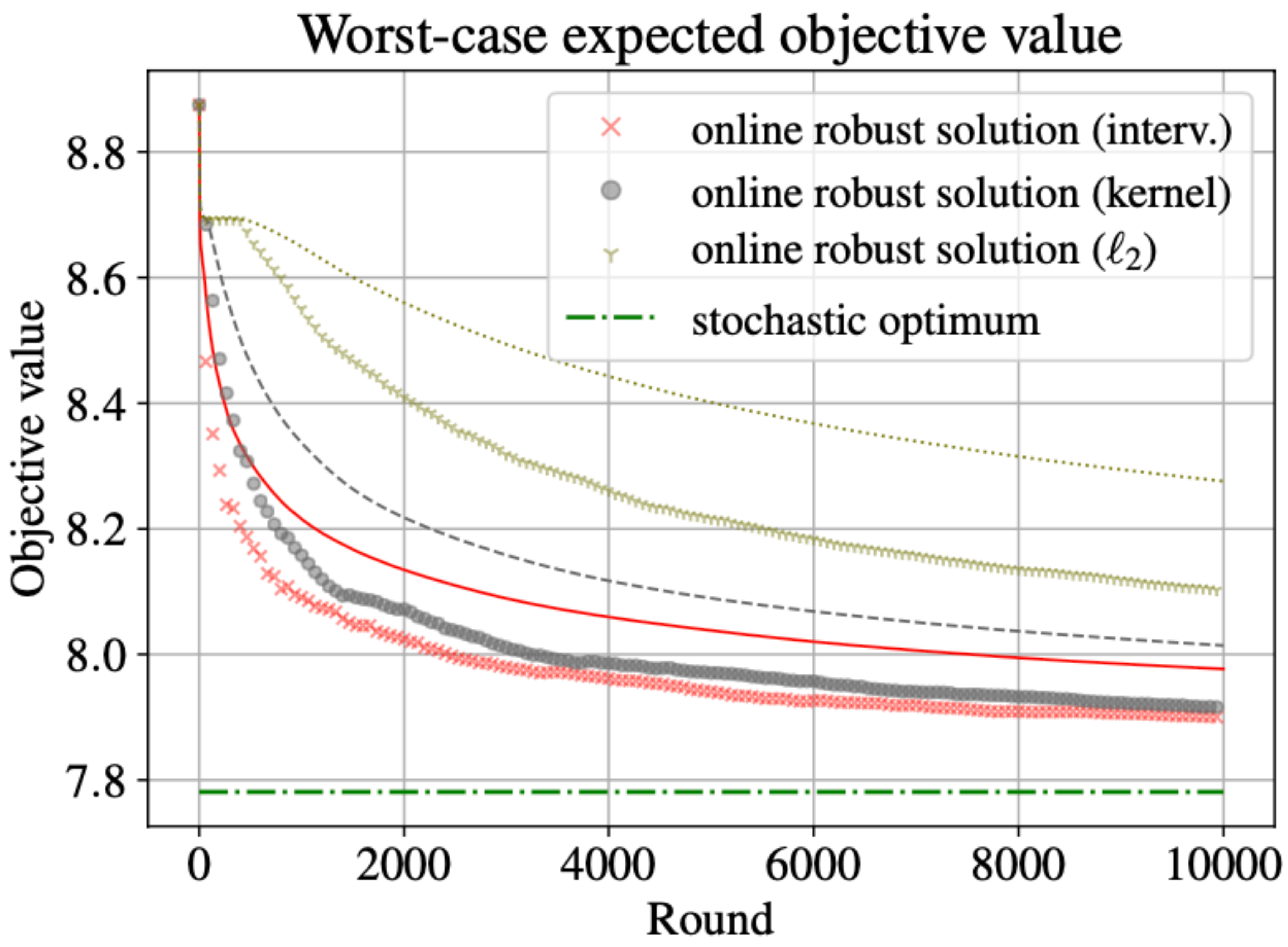

The objective value shrinks for all set types

Fastest reduction for confidence intervals

Benchmark Instances: Running Times

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	$\lvert\mathcal{S}\rvert$	Online Robust	Exact DRO
MIP (I)	10	52.4s	115.8s
MIP (ℓ_2)	10	49.4s	127.5s
MIP (K)	10	56.3s	129.5s
MIP (I)	50	57.7s	176.7s*
MIP (ℓ_2)	50	60.4s	206.1s*
MIP (K)	50	67.0s	244.4s*
MIQP (I)	2	170.2s	271.4s*
MIQP (ℓ_2)	2	186.3s	329.5s*
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More time savings for large and non linear problems

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	MIP $\lvert\mathcal{S}\rvert = 10$	MIP $\lvert\mathcal{S}\rvert = 50$	MIQP $\lvert\mathcal{S}\rvert = 2$
DRO	45.6s	55.9s*	271.4s*
Wassertein	52.3s	59.1s	299.9s
DRBO	42.7s**	66.1s**	738.3s*
Online robust	26.8s	27.1s	170.2s
Running SO	26.6s	26.9s	172.6s

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Online robust methods are significantly faster

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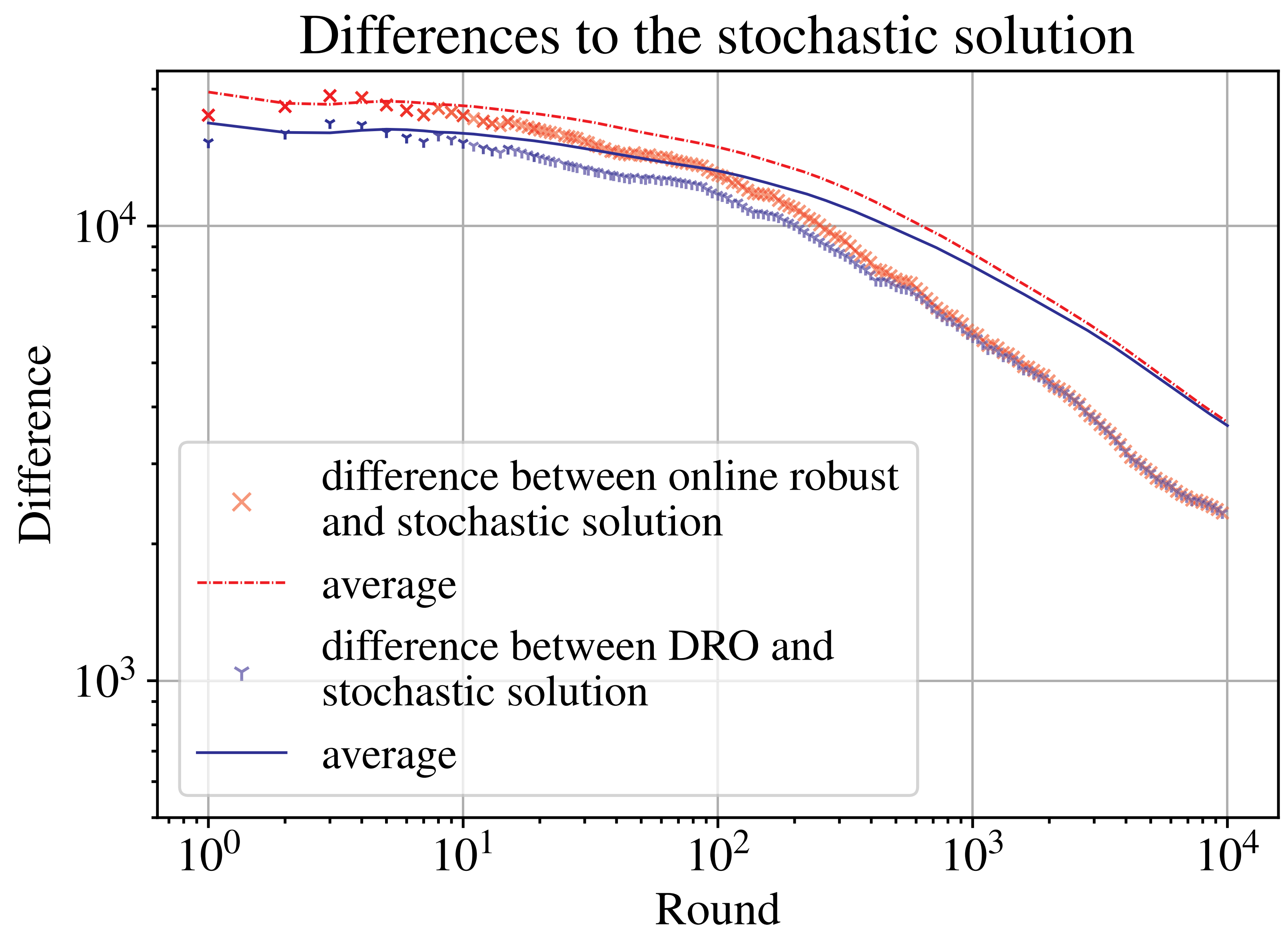
Distributionally Robust Network Design

- Compute minimum cost network topology and edge capacity to satisfy demand
- Demand is uncertain. Interval ambiguity sets.
- **Instances**
  - res8: V = 50, E = 77
  - w1_100: V = 100, E = 207
  - w1_200: V = 200, E = 775

Running Times

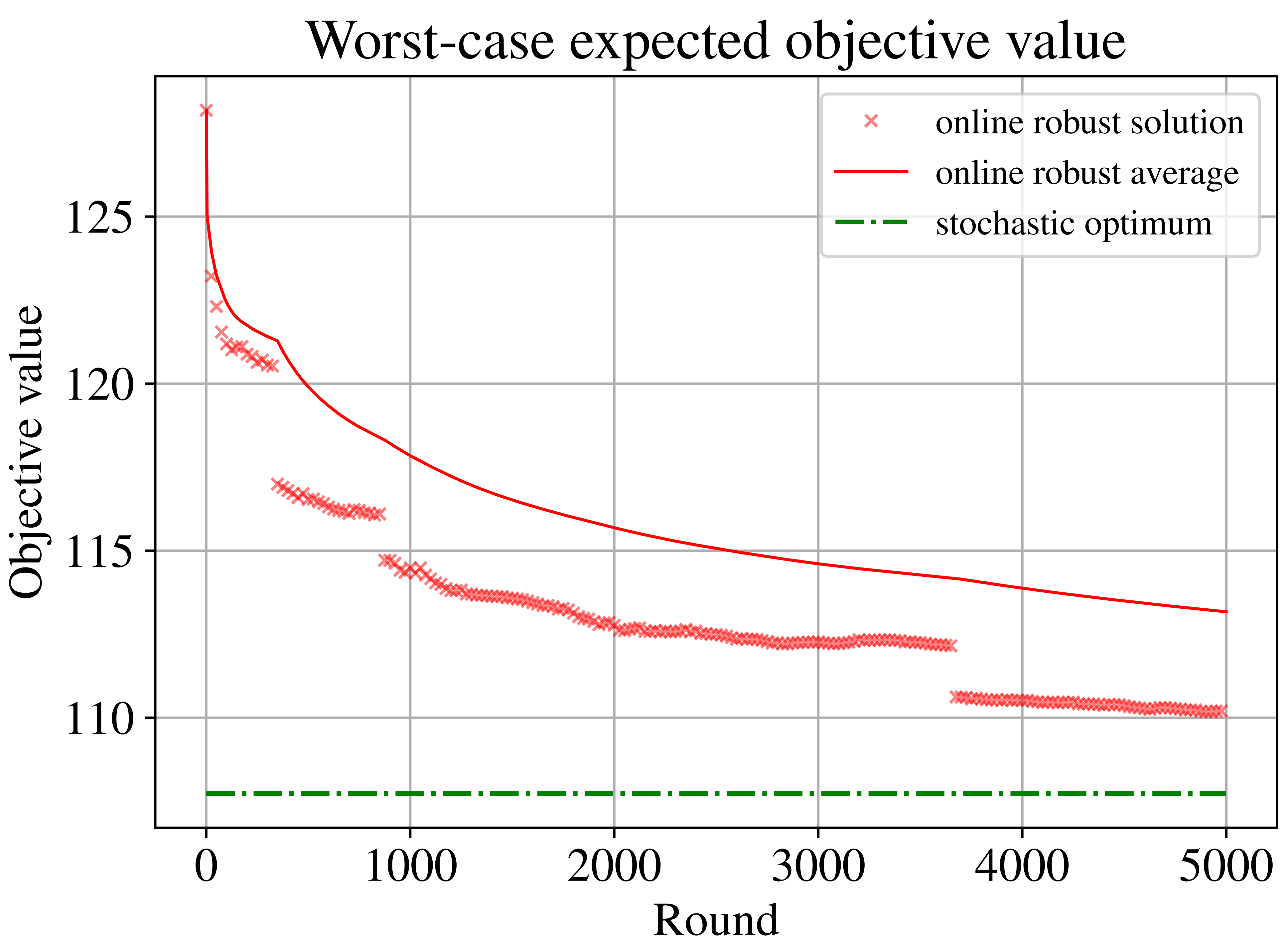
	$\lvert\mathcal{S}\rvert$	Online robust	Exact DRO
res8	10	0.2s	0.5s
res8	50	0.6s	11.6s
w1_100	10	0.3s	32.0s
w1_100	50	1.5s	95.6s
w1_200	10	1.2s	38.7s
w1_200	50	4.7s	1282.2s

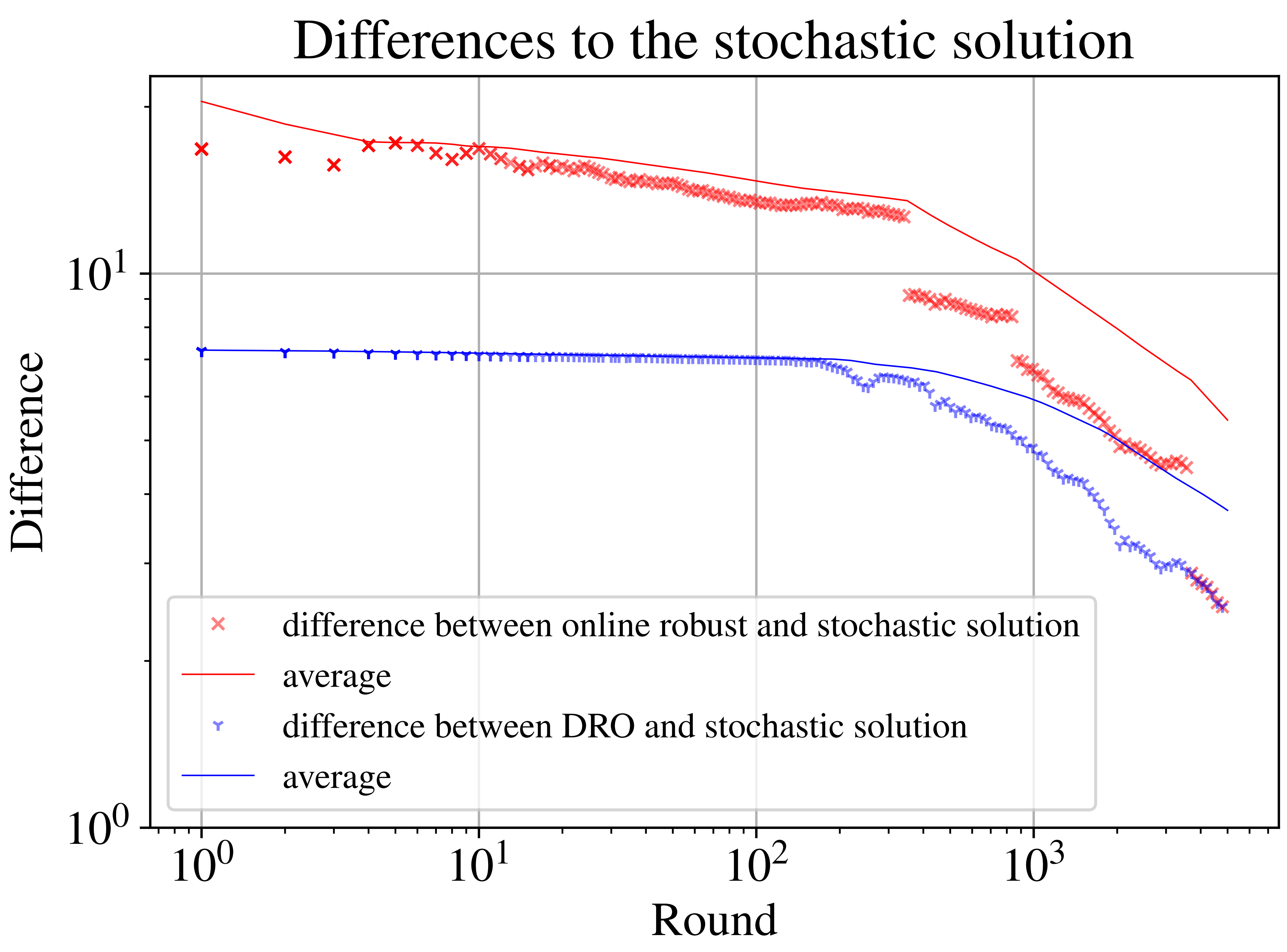


Optimal Route Choice

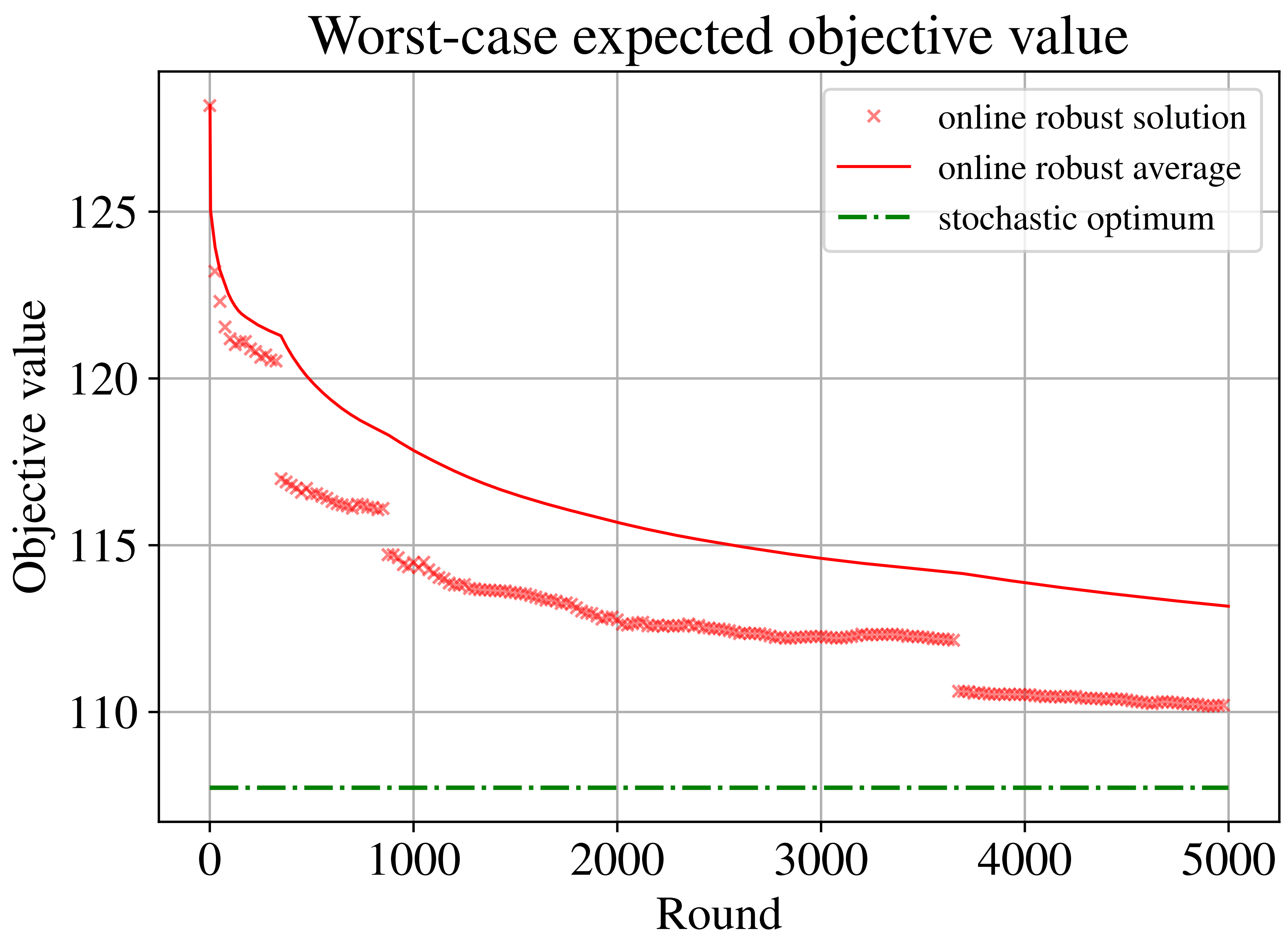
- Choose the shortest paths in a street network with uncertain arc times
- Model: *ChicagoSketch* with 933 nodes and 2950 arcs
- Randomly generated *true* probability distribution for arcs lengths
- Solving directly eliminates structure

Optimal Route Choice

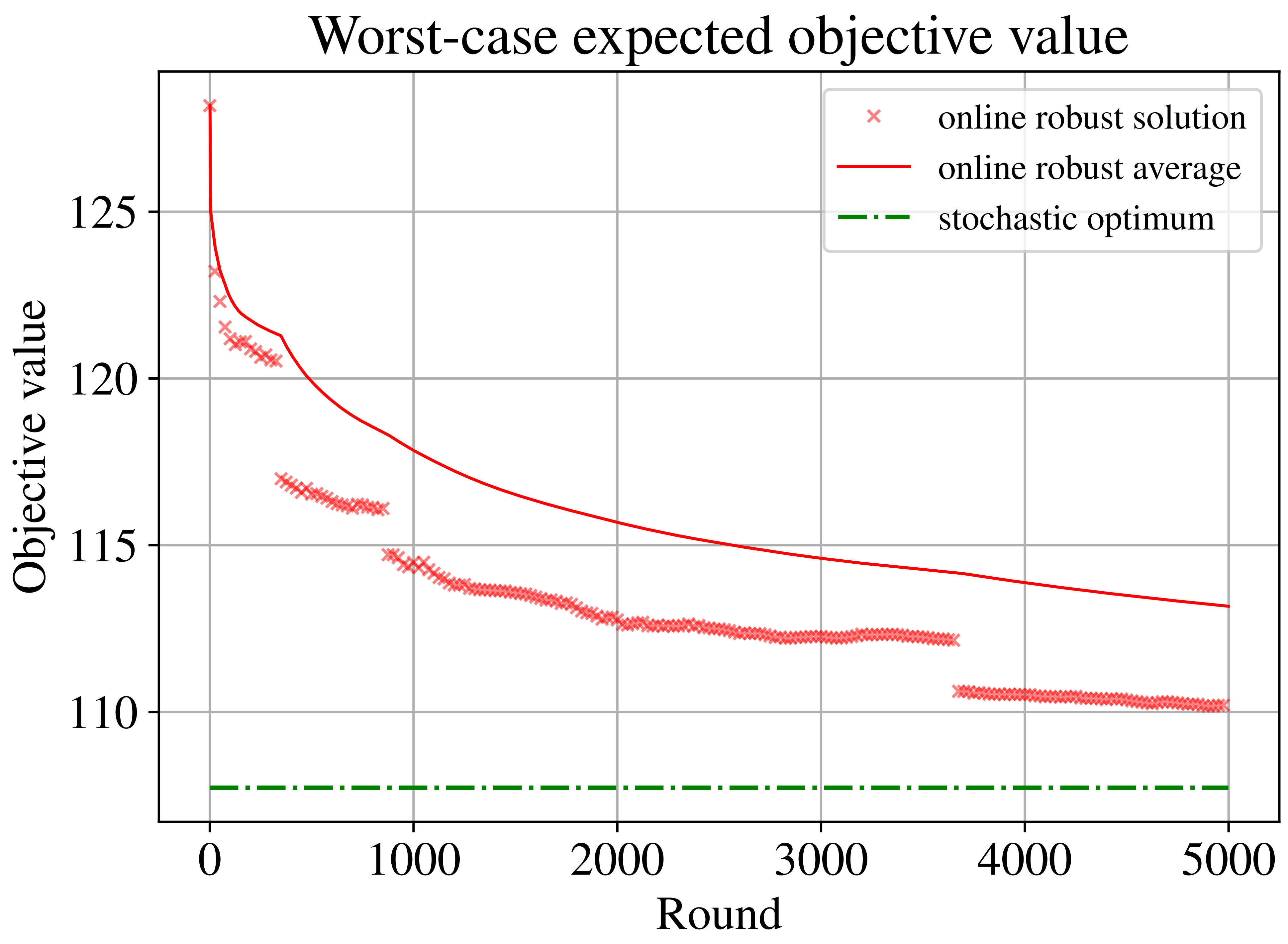


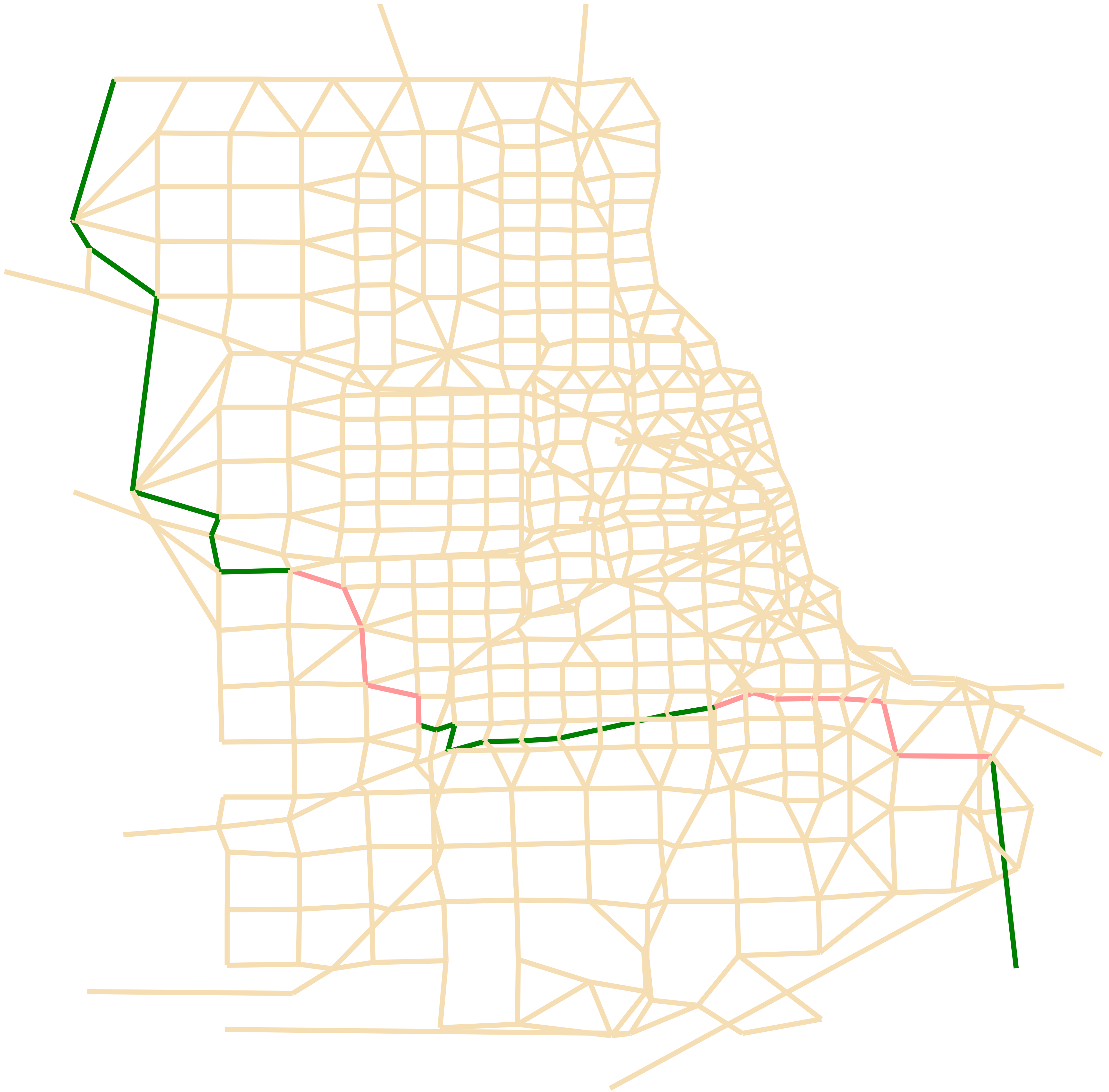


Optimal Route Choice

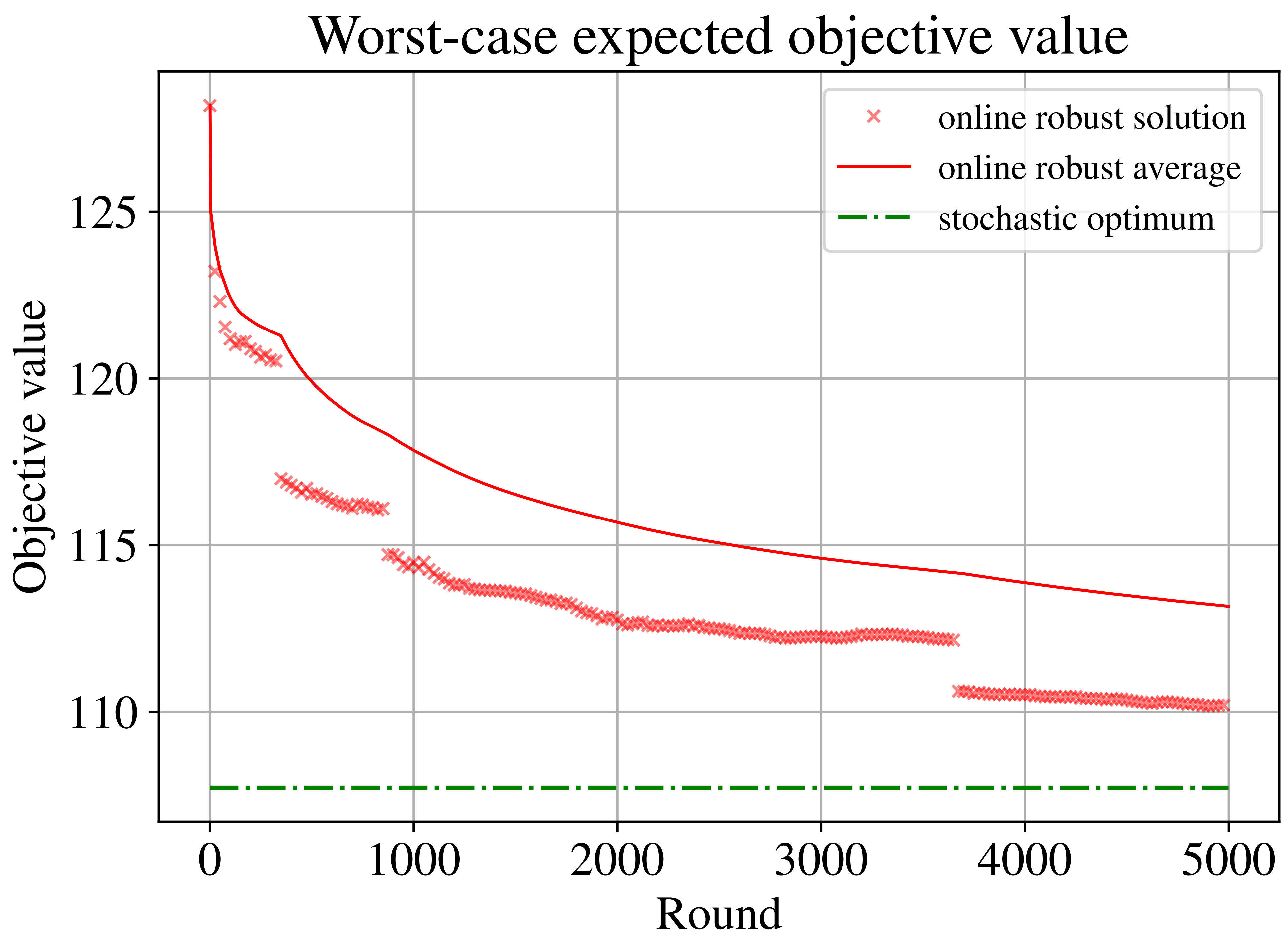


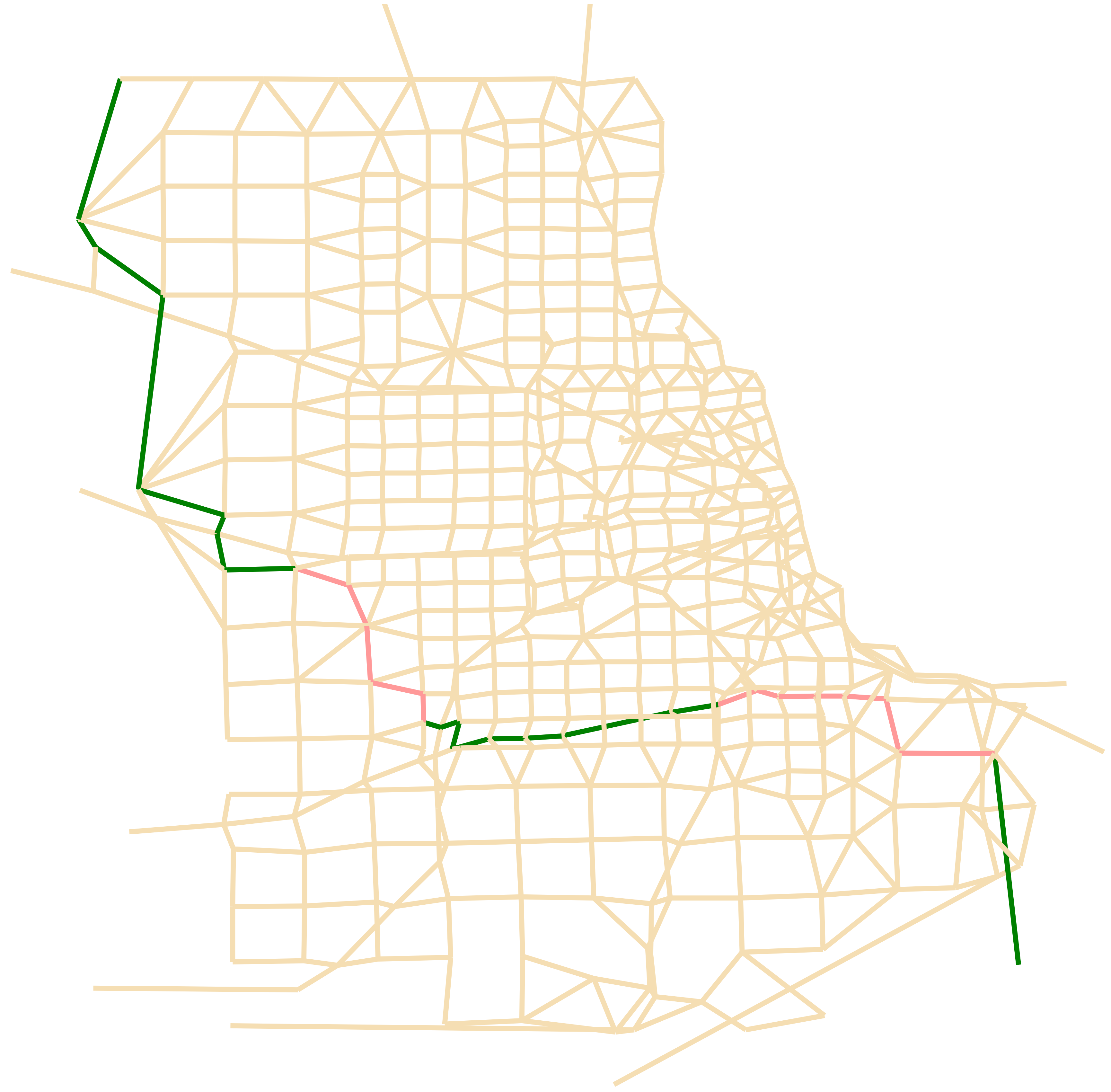
Optimal Route Choice



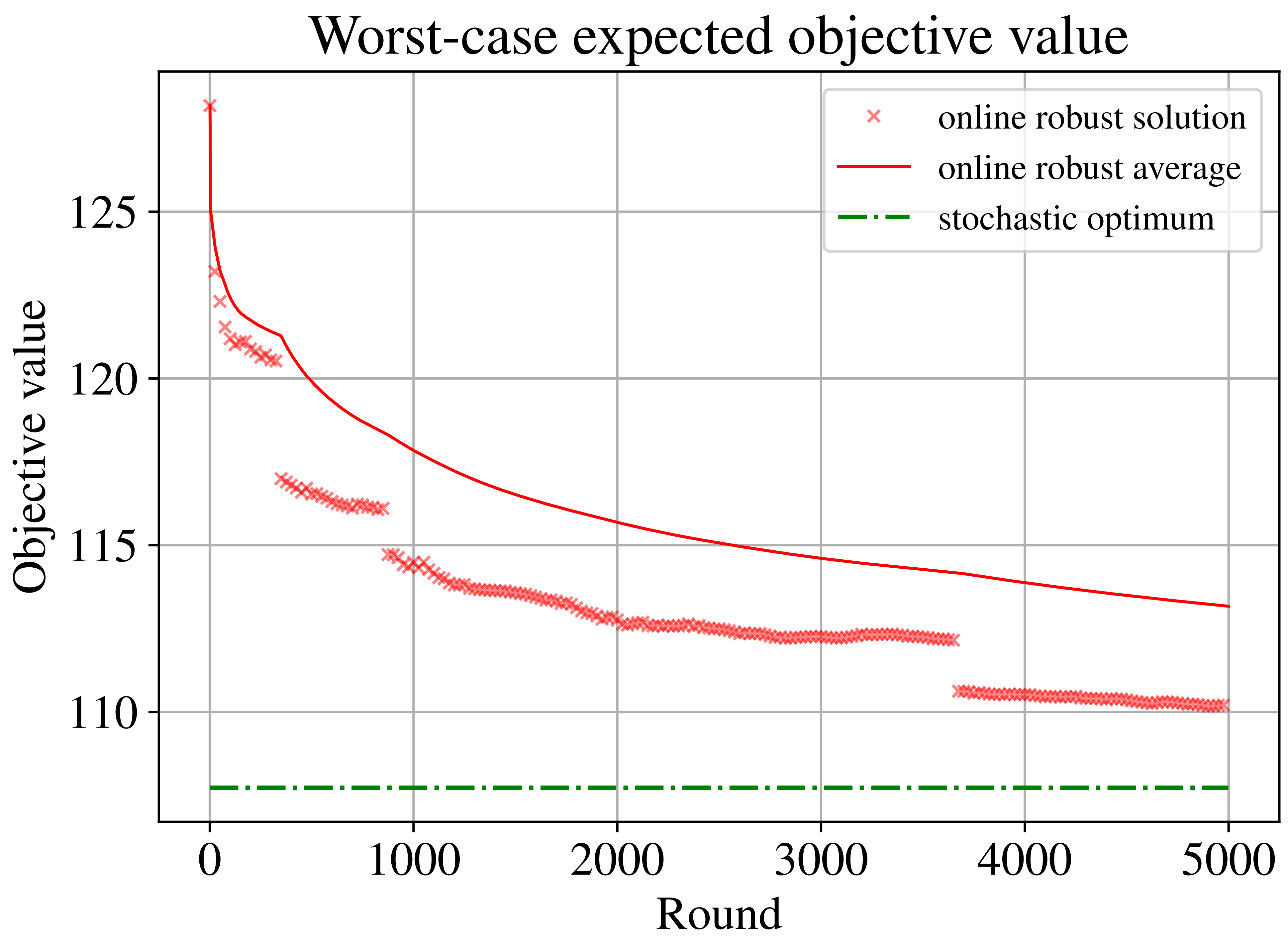


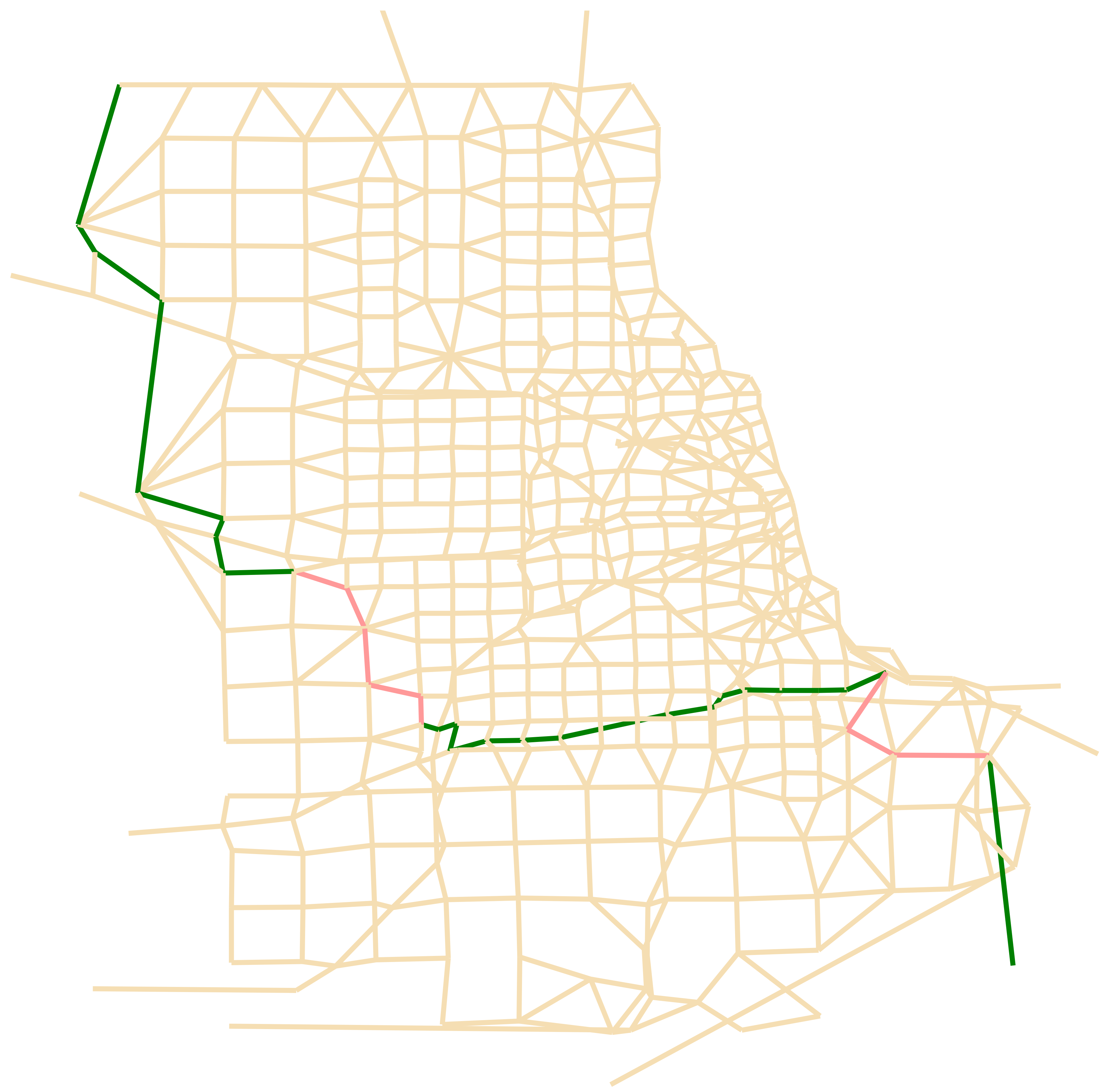
Optimal Route Choice



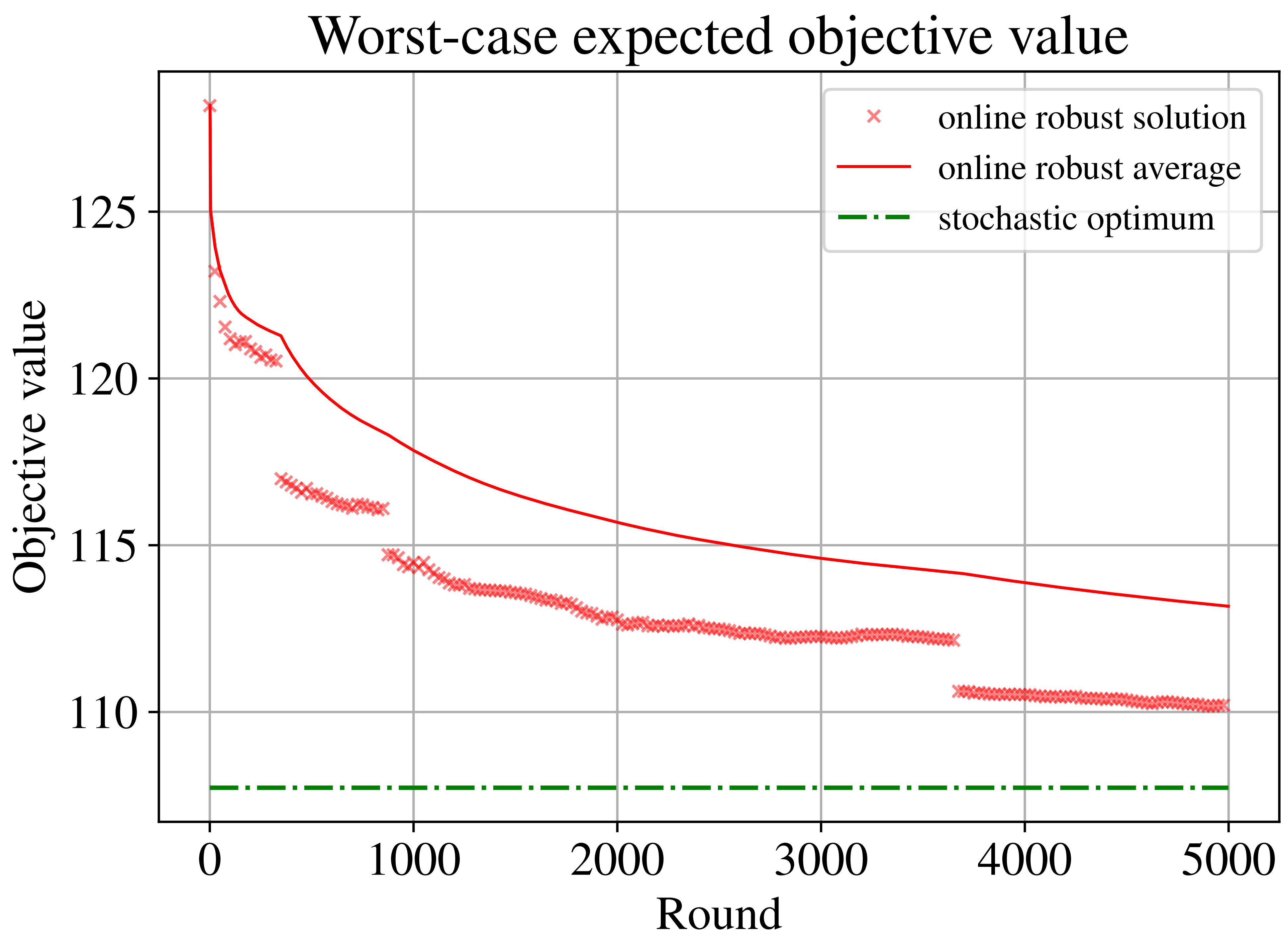


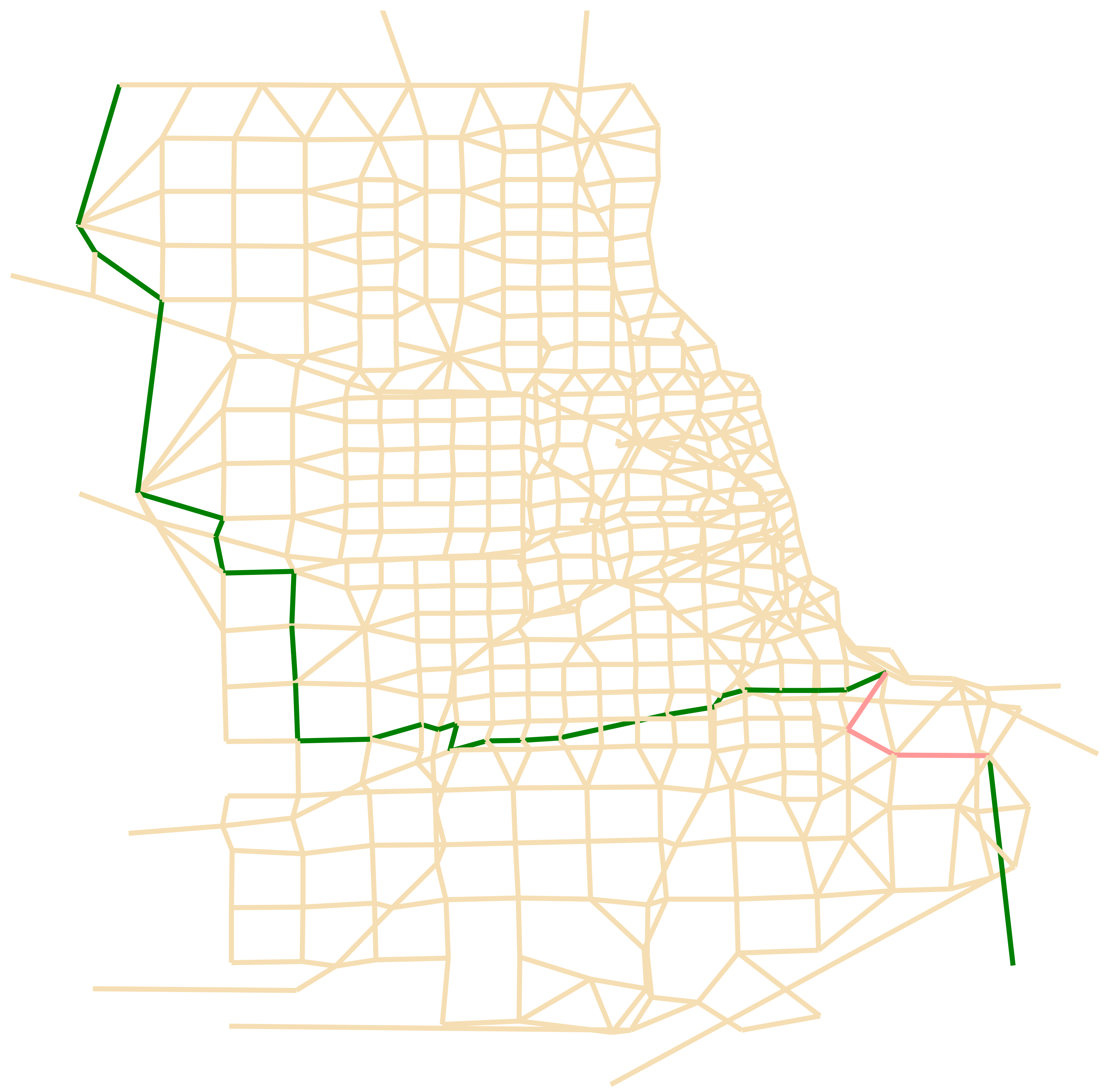
Optimal Route Choice



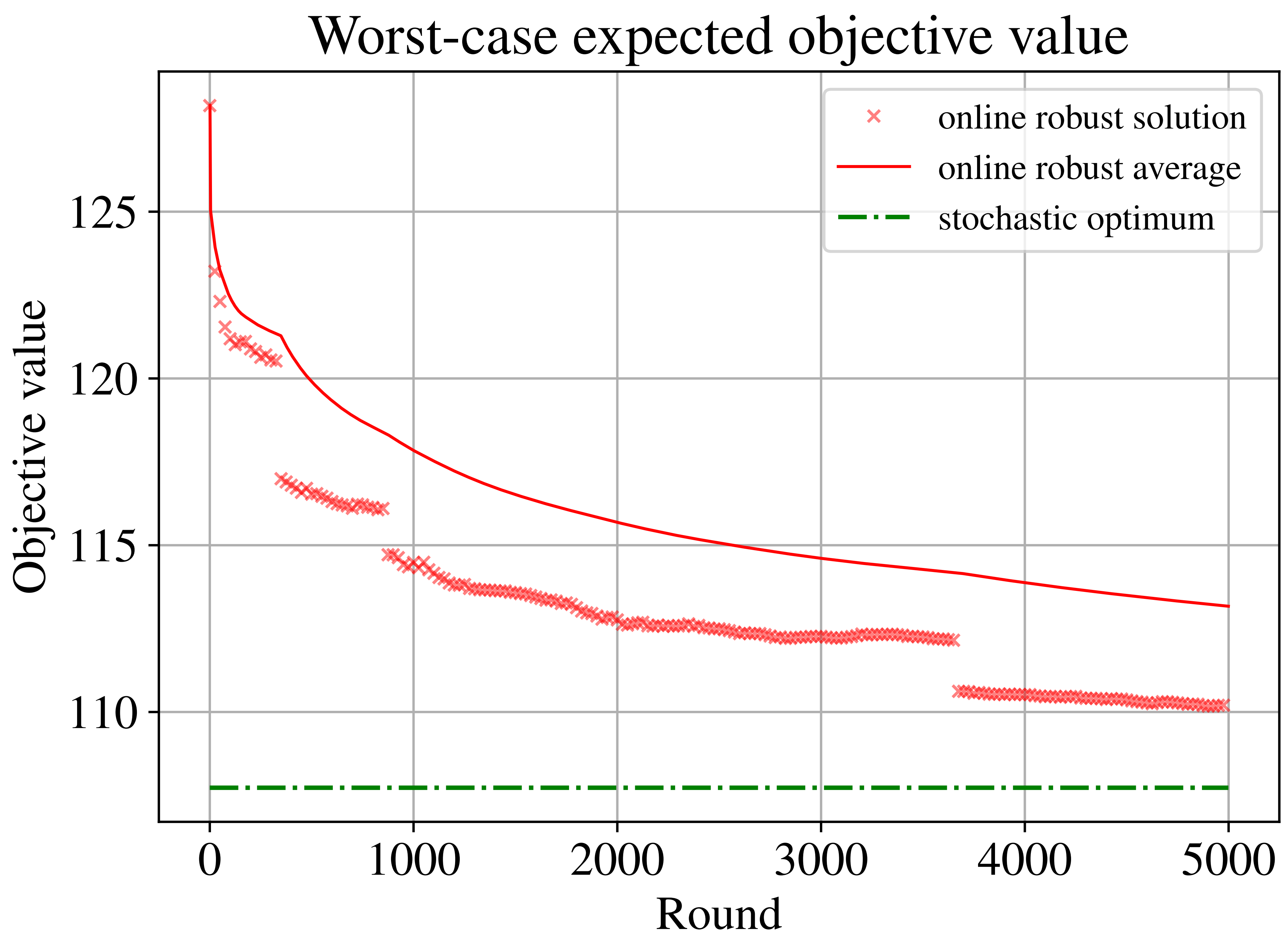


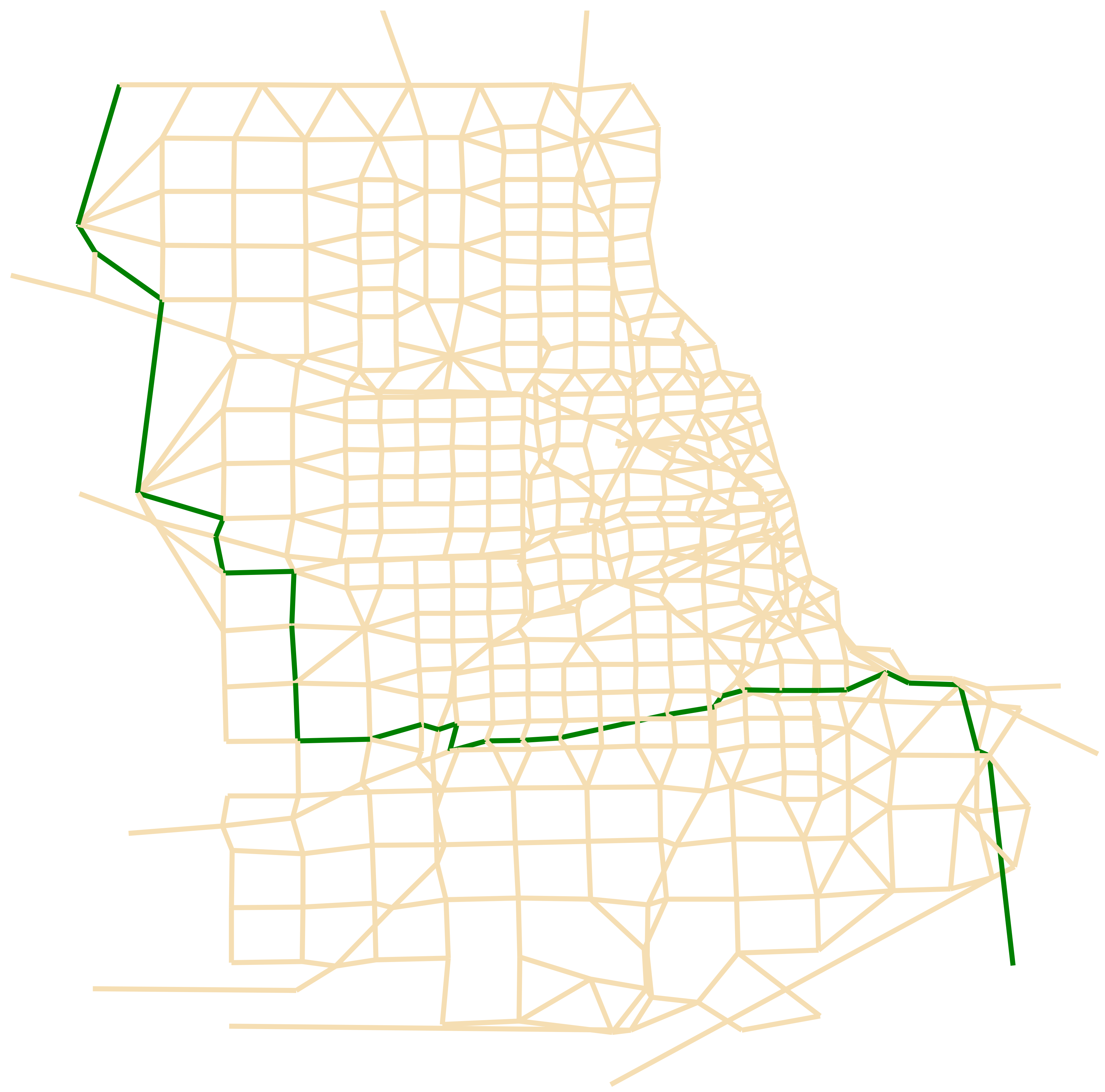
Optimal Route Choice



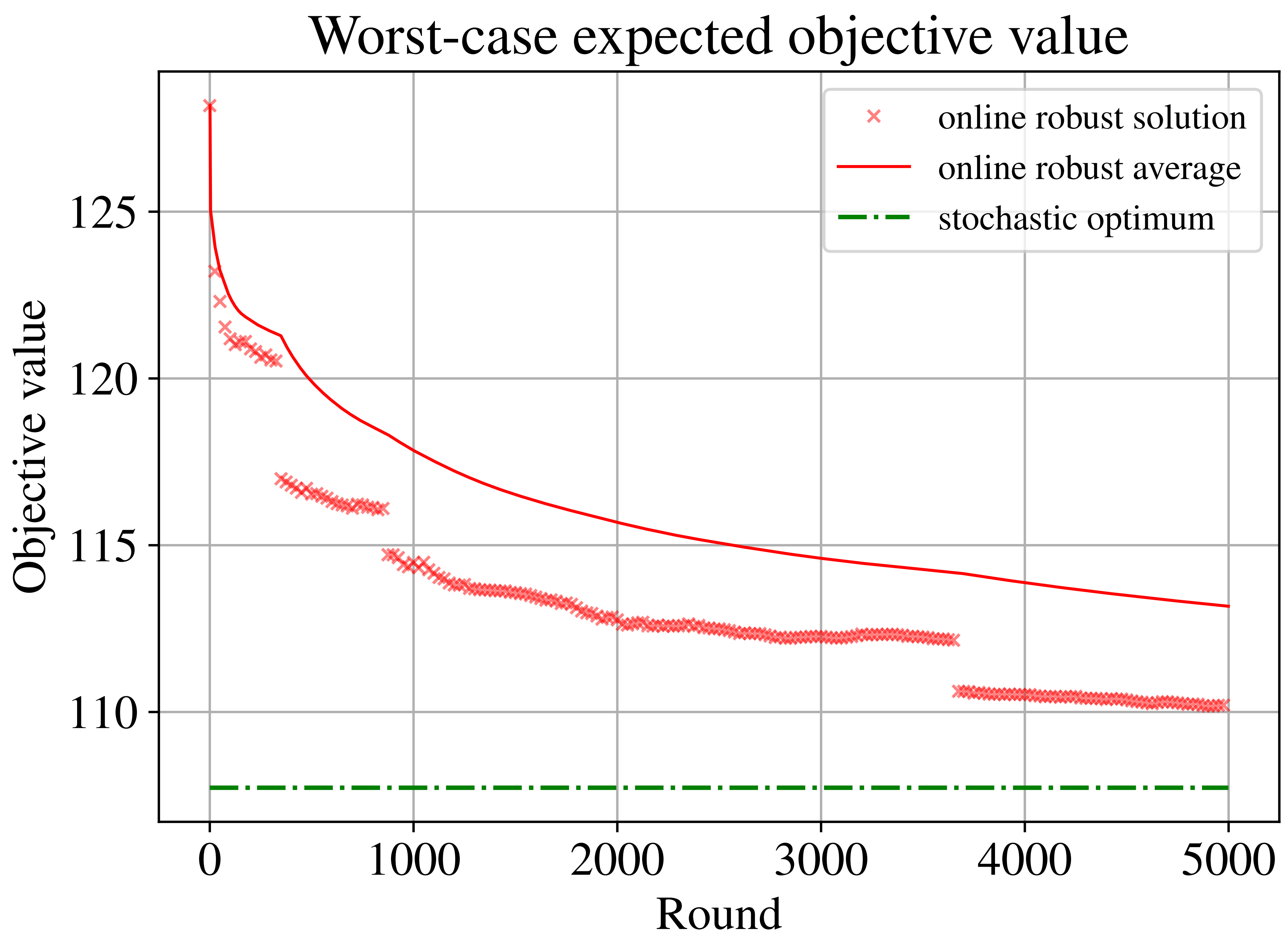


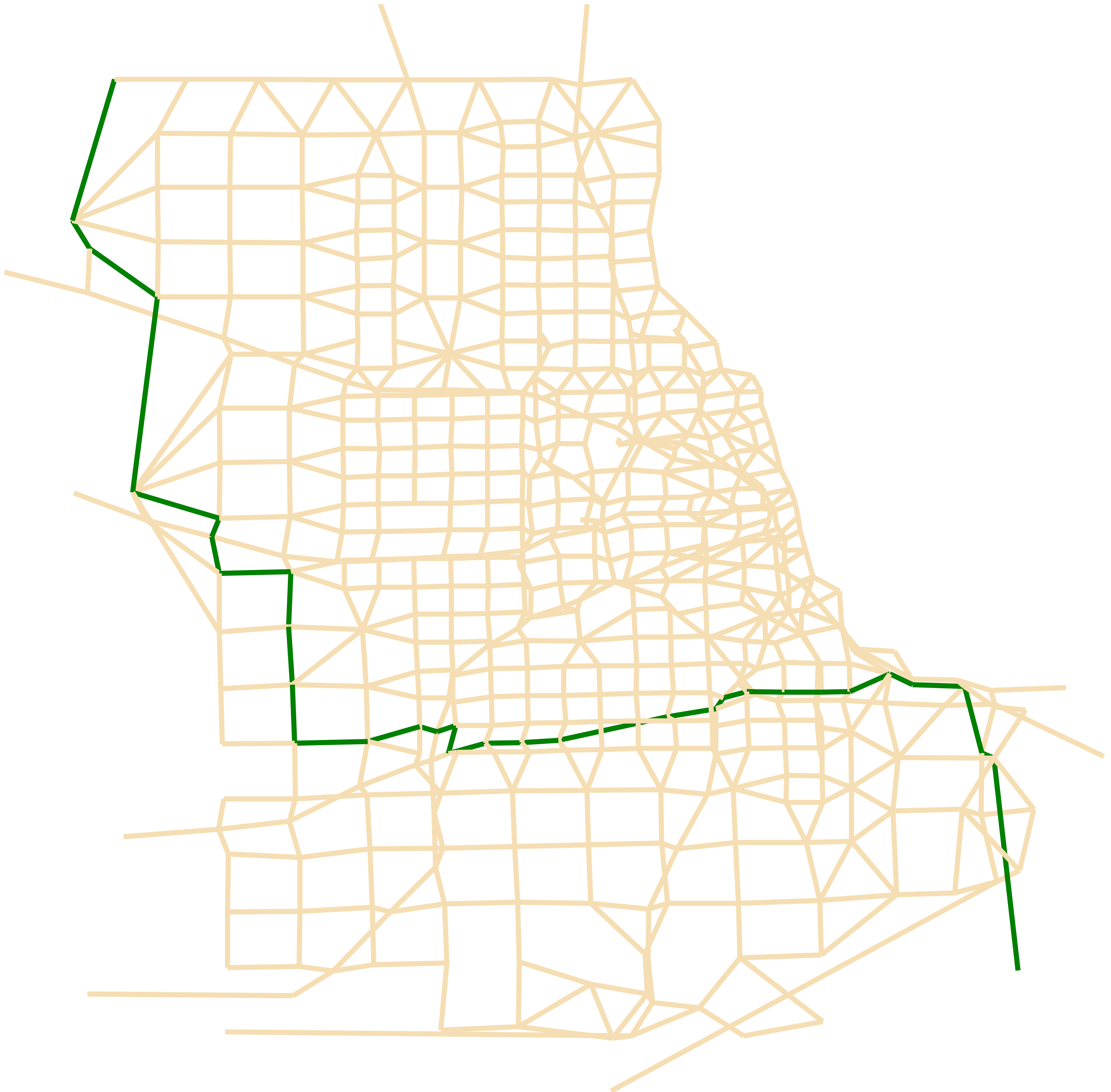
Optimal Route Choice





Optimal Route Choice

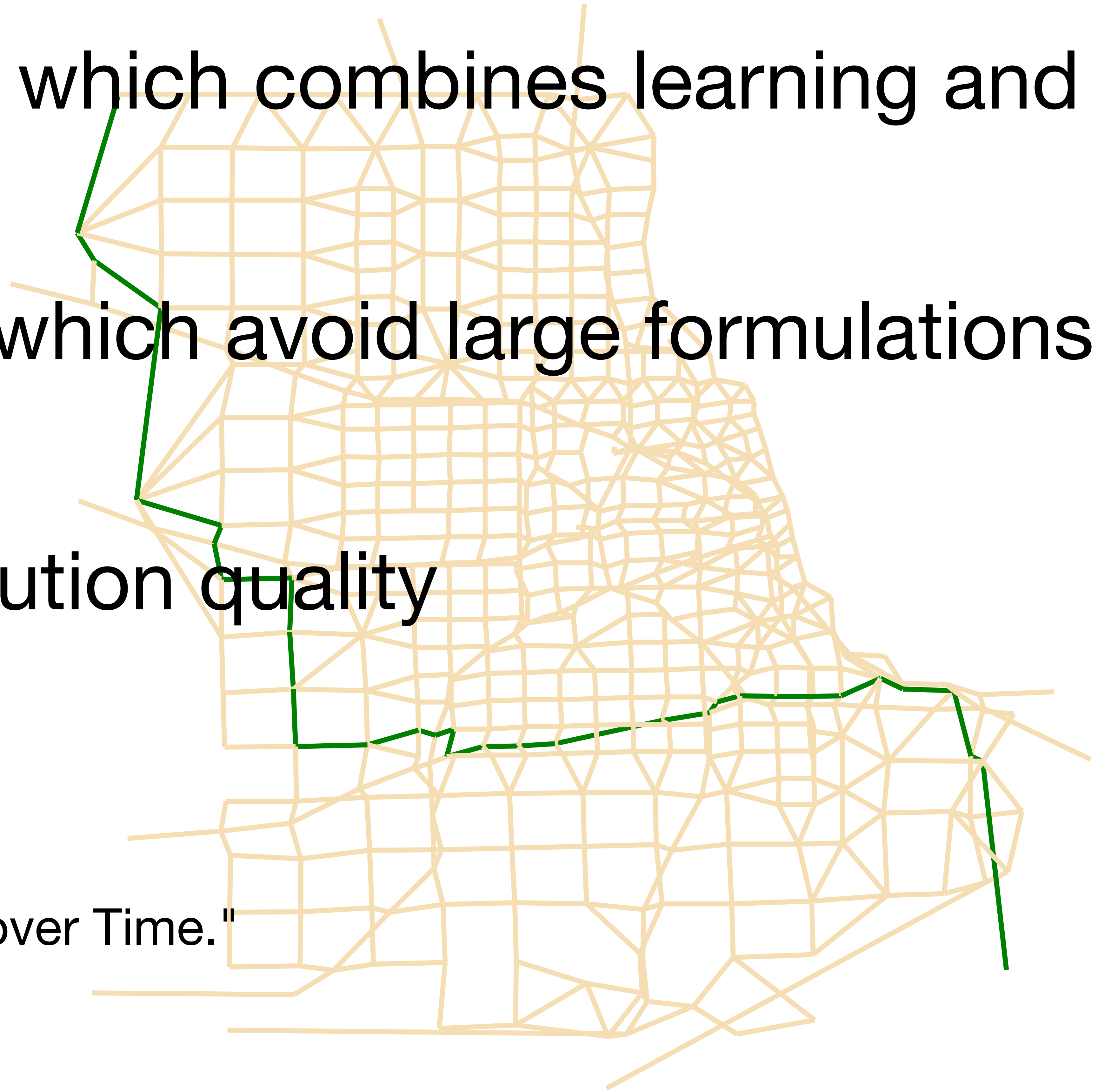




Continuous decrease in objective value but discrete jumps due to changing solution

Conclusions

- Method for optimization under uncertainty which combines learning and Distributionally Robust Optimization
- Iterative algorithm for solution of problem which avoid large formulations and maintains structure
- Theoretical proofs of convergence and solution quality
- Numerical illustrations for the results



Aigner et al. "Data-driven Distributionally Robust Optimization over Time." *INFORMS Journal on Optimization* (2023).

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thank you!