Optimization under Connected Uncertainty

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Motivation

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  - Auto-correlated demand in an inventory management problem
  - Auto-correlated returns in a portfolio optimization problems.
- Robust Optimization across multiple periods: Separate uncertainty sets for each period
  - Does not capture correlation

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where

$$\mathcal{U}_1 = \left\{ d_1 \mid |d_1| \leq \rho \right\} \qquad \mathcal{U}_2(d_1) = \left\{ d_2 \mid |d_2| \leq \sqrt{\rho^2 - d_1^2} \right\}$$

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The parameters of the uncertainty set at each period are a function of past realizations.

Past Work

- **Scenario Tress**:Infanger and Morton [1996], De Queiroz and Morton [2013]
- **RO**: Zhao and Zeng [2012], Jiang et al. [2012], Bertsimas and Vayanos [2015], Lorca and Sun [2015,2017], and Nohadani and Roy [2017].
- **DRO**: Analui and Pflug [2014], Xin and Goldberg [2015]

RO Example

Consider

$$\max_{\mathbf{x}_1,\mathbf{x}_2} \mathbf{c}_1^\top \mathbf{x}_1 + \mathbf{c}_2^\top \mathbf{x}_2$$
$$\text{s.t. } \mathbf{d}_1^\top \mathbf{x}_1 + \mathbf{d}_2^\top \mathbf{x}_2 \leq B \quad \forall \mathbf{d}_2 \in \mathcal{U}_2(\mathbf{d}_1) \quad \forall \mathbf{d}_1 \in \mathcal{U}_1$$
$$\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}$$

Uncertainty for $\mathbf{d}_2$ explicitly depends on $\mathbf{d}_1$.

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Uncertainty for $\mathbf{d}_2$ explicitly depends on $\mathbf{d}_1$.

- Ellipsoid

$$\mathcal{U}_2(\mathbf{d}_1) = \{\mathbf{d}_2 \mid \mathbf{d}_2 = \boldsymbol{\mu}_2(\mathbf{d}_1) + \mathbf{L}_2\mathbf{u}_2 : \|\mathbf{u}_2\|_2 \leq r_2\}$$

RO Example

Consider

$$\begin{aligned} \max_{\mathbf{x}_1,\mathbf{x}_2} \ & \mathbf{c}_1^\top \mathbf{x}_1 + \mathbf{c}_2^\top \mathbf{x}_2 \\ \text{s.t.} \ & \mathbf{d}_1^\top \mathbf{x}_1 + \mathbf{d}_2^\top \mathbf{x}_2 \le B \ \ \forall \mathbf{d}_2 \in \mathcal{U}_2(\mathbf{d}_1) \ \ \forall \mathbf{d}_1 \in \mathcal{U}_1 \\ & \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X} \end{aligned}$$

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- Center

$$\boldsymbol{\mu}_2(\mathbf{d}_1) = \mathbf{A}_2\boldsymbol{\mu}_1(\mathbf{d}_0) + \mathbf{F}_2\mathbf{d}_1 + \mathbf{c}_2$$

DRO Example

$$\sup_{P_1 \in \widetilde{\mathcal{U}}_1} \mathbb{E}_{\mathbf{d_1} \sim P_1} \left[\mathbf{d}_1^\top \mathbf{x}_1 \right] \leq B. \tag{DRO}$$

DRO Example

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$$\widetilde{\mathcal{U}}_1 = \left\{ P_1 \in \mathcal{M}(\Xi_1) \middle| \; \left| \mathbb{E}_{P_1}[\mathbf{d}_1] - \boldsymbol{\mu}_1 \right| \leq \boldsymbol{\delta}_1, \; \mathbb{E}_{P_1}[(\mathbf{d}_1 - \boldsymbol{\mu}_1)(\mathbf{d}_1 - \boldsymbol{\mu}_1)^\top] \preceq \boldsymbol{\Sigma}_1 \right\},$$

DRO Example

$$\sup_{P_1 \in \widetilde{\mathcal{U}}_1} \mathbb{E}_{\mathbf{d}_1 \sim P_1} \left[\mathbf{d}_1^\top \mathbf{x}_1 + \sup_{P_2 \in \widetilde{\mathcal{U}}_2(\mathbf{d}_1)} \mathbb{E}_{\mathbf{d}_2 \sim P_2} \left[\mathbf{d}_2^\top \mathbf{x}_2 \right] \right] \leq B. \quad \text{(DRO)}$$

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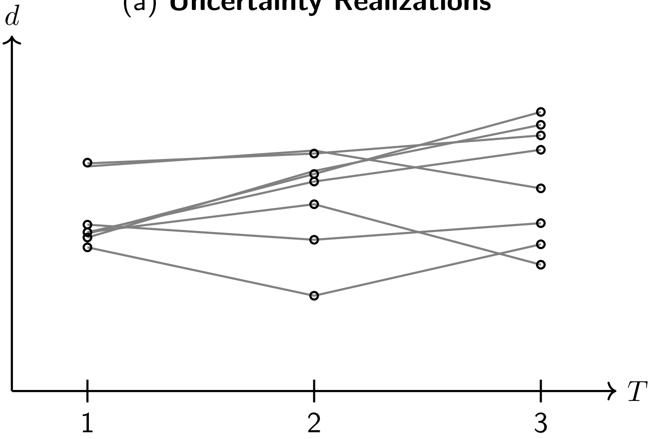
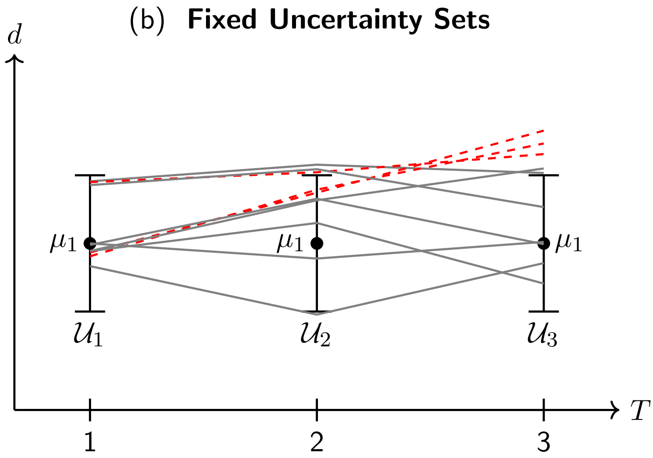
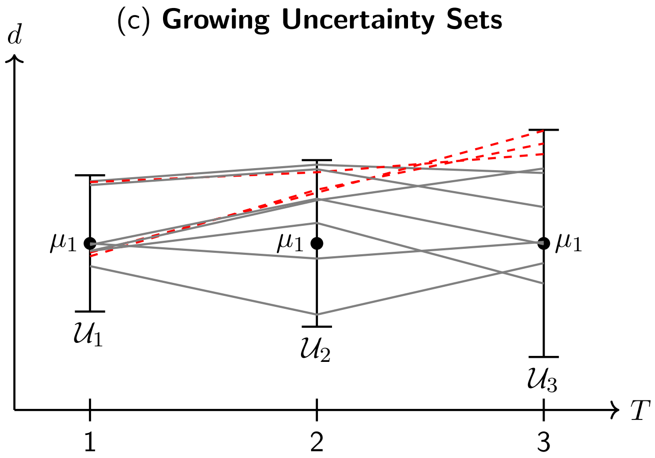
$$\widetilde{\mathcal{U}}_2(\mathbf{d}_1) = \left\{ P_{2|1} \in \mathcal{M}(\Xi_2) \middle| \ \left| \mathbb{E}_{P_{2|1}}[\mathbf{d}_2] - \boldsymbol{\mu}_2(\mathbf{d}_1) \right| \leq \boldsymbol{\delta}_2, \ \mathbb{E}_{P_{2|1}}[(\mathbf{d}_2 - \boldsymbol{\mu}_2^0)(\mathbf{d}_2 - \boldsymbol{\mu}_2^0)^\top] \preceq \boldsymbol{\Sigma}_2 \right\}$$

Benefits of CU Sets

(a) **Uncertainty Realizations**

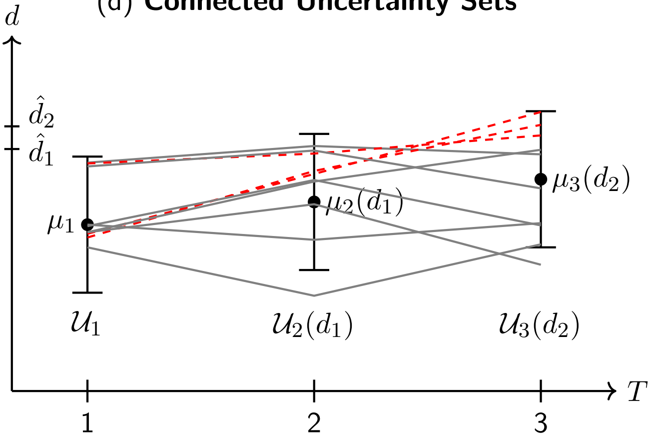



(b) **Fixed Uncertainty Sets**
d
μ_1
μ_1
μ_1
$\mathcal{U}_1$
$\mathcal{U}_2$
$\mathcal{U}_3$
T
1
2
3

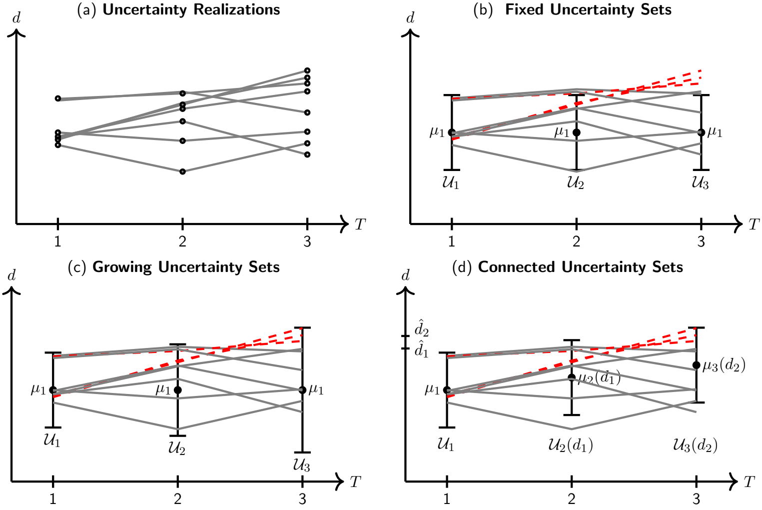

(c) Growing Uncertainty Sets
d
T
μ_1
μ_1
μ_1
$\mathcal{U}_1$
$\mathcal{U}_2$
$\mathcal{U}_3$
1
2
3

Benefits of CU Sets

(d) **Connected Uncertainty Sets**



Benefits of CU Sets



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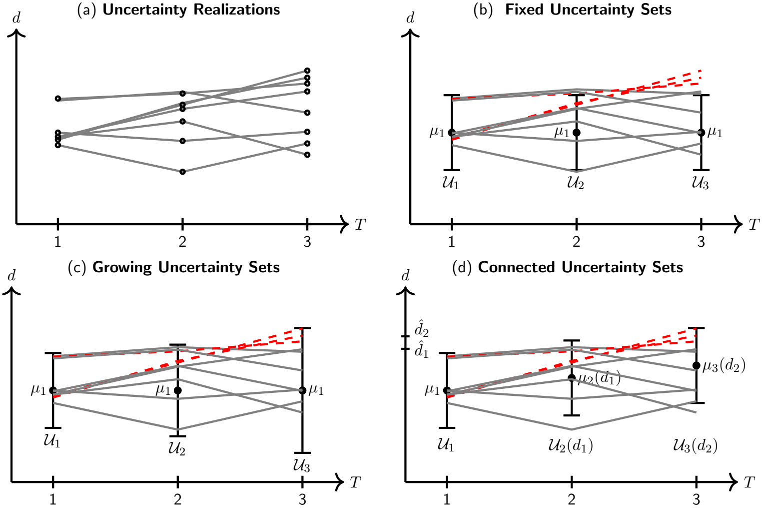
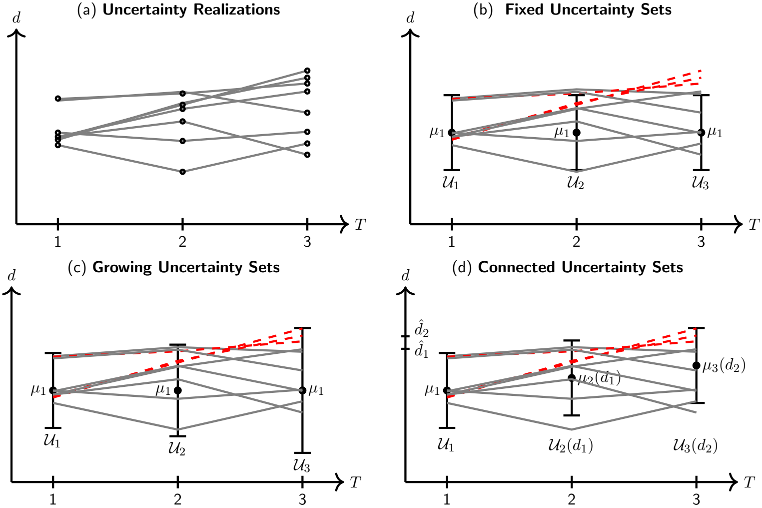


- Provides better coverage of uncertainty realizations

Benefits of CU Sets



- Provides better coverage of uncertainty realizations
- Incorporates information about structure of uncertainty realizations.

DRO: Insights

$$\mathbb{E}\left[\sum_{t=1}^{T} h_t(\mathbf{x}_t, \mathbf{d}_t)\right] \leq B$$

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- Robust counterpart

$$\sup_{P_1 \in \widetilde{\mathcal{U}}_1} \mathbb{E}_{P_1}\left[h_1 + \sup_{P_{2|1} \in \widetilde{\mathcal{U}}_2(\mathbf{d}_1)}\left\{\mathbb{E}_{P_{2|1}}\left[h_2 + \cdots + \sup_{P_{T|T-1} \in \widetilde{\mathcal{U}}_T(\mathbf{d}_{T-1})}\left\{\mathbb{E}_{P_{T|T-1}}\left[h_T\right]\right\}\right]\right\}\right] \leq B.$$

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- Each nested $\sup$ is a concave function $\mathbf{d}_t$.

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- Each nested $\sup$ is a concave function $\mathbf{d}_t$.
- Reformulate as infinite dimensional optimization problem.

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- IOP

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- IOP $\xrightarrow{\text{conser. approx.}}$ Semi-infinite opt. prob. $\xrightarrow{\text{ellips. support}}$ SDP
- Conserv. Approx. $\Leftrightarrow$ approximating $\sup$ by a linear function

Portfolio Optimization: Model

- Two period portfolio optimization problem

$$
\begin{aligned}
\max_{x_1, x_2} \; \min_{P \in \mathcal{U}} \; & \mathbb{E}_P[u(\mathbf{r}_1^\top \mathbf{x}_1) + u(\mathbf{r}_2^\top \mathbf{x}_2)] \\
\text{s.t.} \; & \sum_{i=1}^{n} x_{1i} = 1 \\
& \sum_{i=1}^{n} x_{2i} = 1 \\
& \mathbf{x}_1, \mathbf{x}_2 \geq 0
\end{aligned}
$$

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- Maximize $u(z) = \min\{1.5z, 0.015 + z, 0.06 + 0.2z\}$

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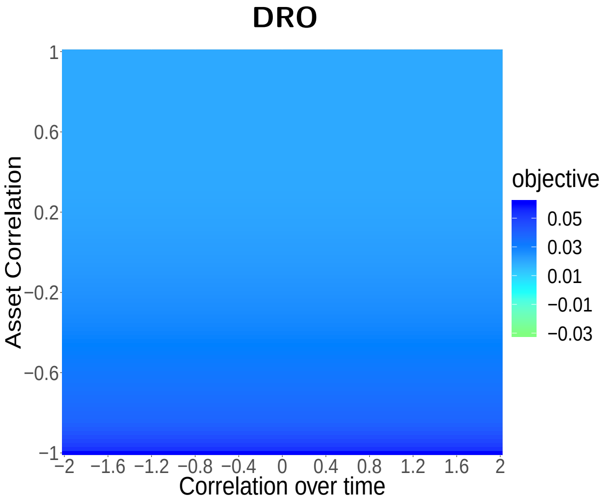
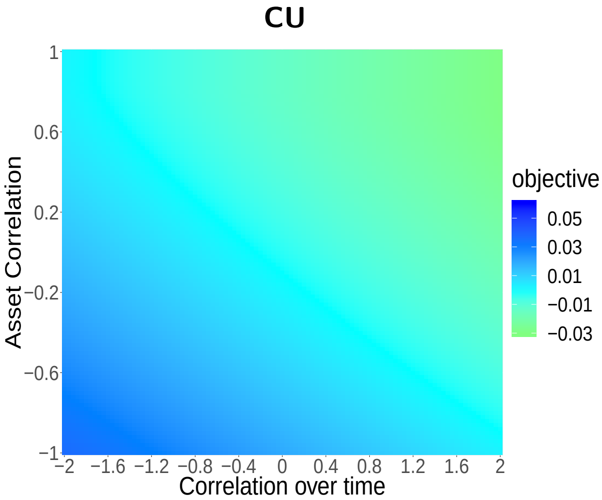
- 2 Stocks
- Moment based uncertainty set. $\boldsymbol{\mu}_2(\mathbf{r}_1) = \omega\mathbf{r}_1 + (1-\omega)\boldsymbol{\mu}_1$
- $\boldsymbol{\mu}_1 = [0.06, 0.03], \sigma^2 = 0.005$ for both assets

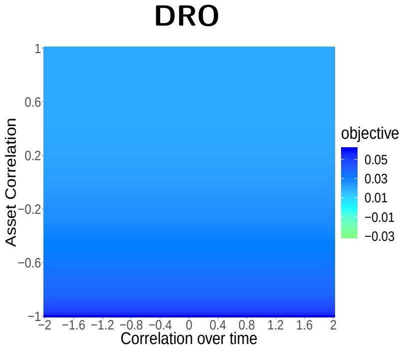
Portfolio Optimization: Objective



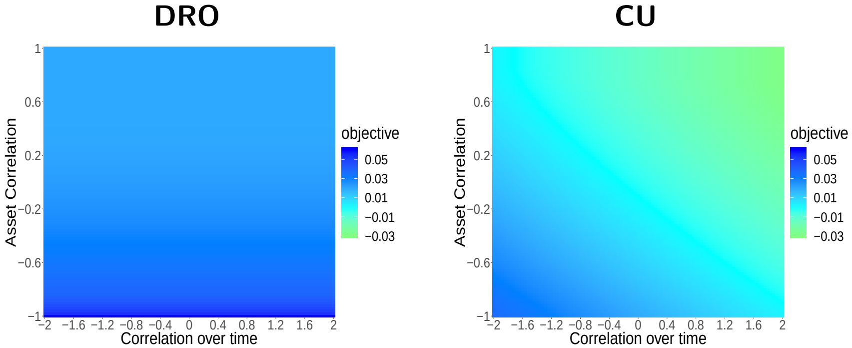
Portfolio Optimization: Objective



Portfolio Optimization: Objective

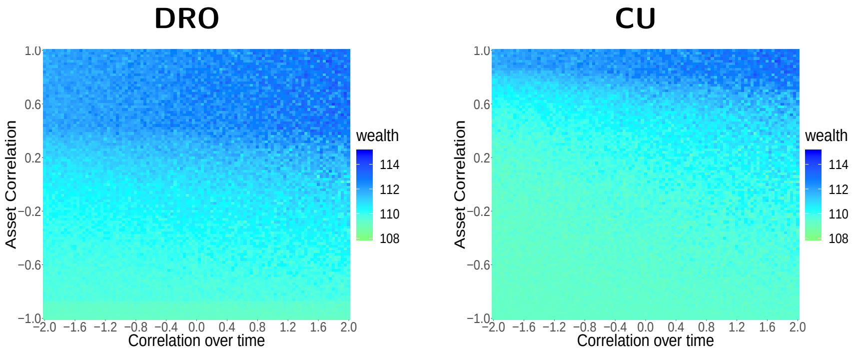


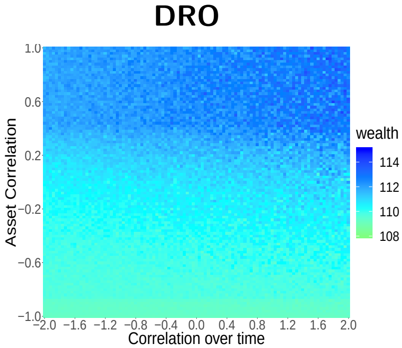
Portfolio Optimization: Objective

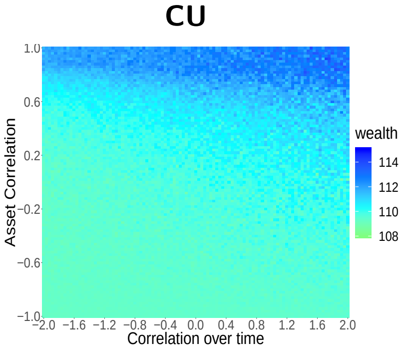


Objective value decreases with asset correlation and correlation over time.

Portfolio Optimization: Realized Wealth

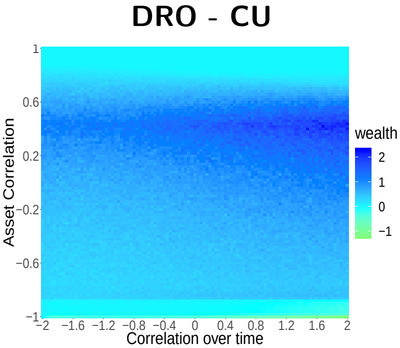




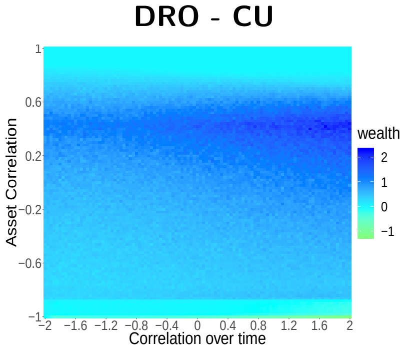


Average realized wealth increases with asset correlation and correlation over time.

Portfolio Optimization: Wealth Standard Deviation



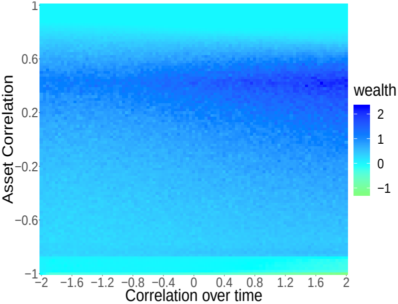
Portfolio Optimization: Wealth Standard Deviation



Benefit of CU increases with correlation over time. It achieves a peak for an asset correlation value 0.4-0.5

Conclusions

- In many problems, uncertainty depends on past realizations.
- Connected uncertainty sets incorporate this behavior and lead to less variable solutions.
- The reformulated problems are difficult → decision rules and conservative approximations.



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RO: Insights

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- Linear constraints with polyhedral or ellipsoidal uncertainty sets are reformulated as LPs and SOCPs.

Robust Knapsack

- Two period knapsack problem with uncertain weight coefficients.

$$
\begin{aligned}
\min_{\mathbf{x}_1,\mathbf{x}_2}\ & \mathbf{c}_1^\top \mathbf{x}_1 + \mathbf{c}_2^\top \mathbf{x}_2 \\
\text{s.t. } & \mathbf{d}_1^\top \mathbf{x}_1 + \mathbf{d}_2^\top \mathbf{x}_2 \leq B \ \ \forall \mathbf{d}_2 \in \mathcal{U}_2(\mathbf{d}_1) \ \ \forall \mathbf{d}_1 \in \mathcal{U}_1 \\
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- Ellipsoidal uncertainty sets. Parameters from samples.

Robust Knapsack

- Two period knapsack problem with uncertain weight coefficients.

$$\begin{aligned} \min_{\mathbf{x}_1, \mathbf{x}_2} \; & \mathbf{c}_1^\top \mathbf{x}_1 + \mathbf{c}_2^\top \mathbf{x}_2 \\ \text{s.t. } & \mathbf{d}_1^\top \mathbf{x}_1 + \mathbf{d}_2^\top \mathbf{x}_2 \le B \;\; \forall \mathbf{d}_2 \in \mathcal{U}_2(\mathbf{d}_1) \;\; \forall \mathbf{d}_1 \in \mathcal{U}_1 \\ & \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X} \end{aligned}$$

- Ellipsoidal uncertainty sets. Parameters from samples.
- Solution quality evaluated on new uncertainty samples.

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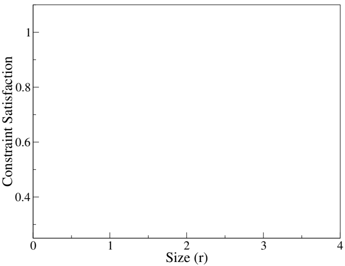
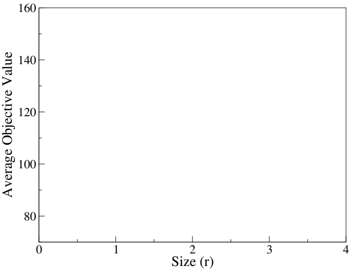
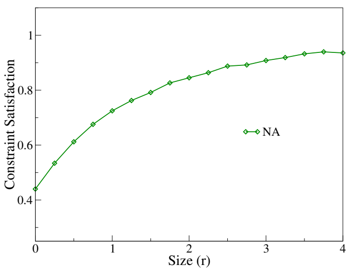
$$
\begin{aligned}
\min_{\mathbf{x}_1,\mathbf{x}_2} \ & \mathbf{c}_1^\top \mathbf{x}_1 + \mathbf{c}_2^\top \mathbf{x}_2 \\
\text{s.t. } & \mathbf{d}_1^\top \mathbf{x}_1 + \mathbf{d}_2^\top \mathbf{x}_2 \leq B \ \ \forall \mathbf{d}_2 \in \mathcal{U}_2(\mathbf{d}_1) \ \ \forall \mathbf{d}_1 \in \mathcal{U}_1 \\
& \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}
\end{aligned}
$$

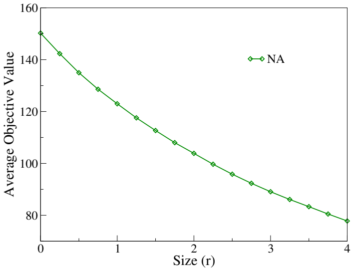
- Ellipsoidal uncertainty sets. Parameters from samples.
- Solution quality evaluated on new uncertainty samples.
- Performance evaluated with Normal and Lognormal distributions.





- Better constraint satisfaction at the price of lower objective value.





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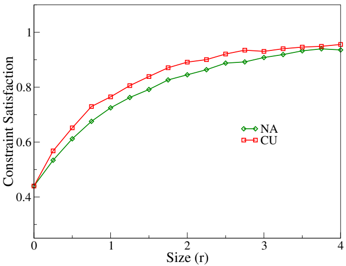

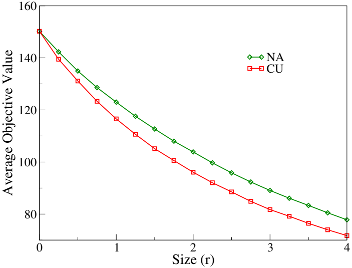

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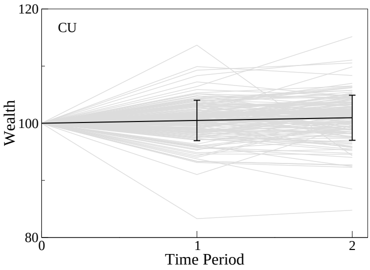
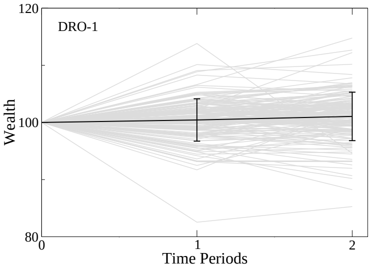

CU
Wealth
120
100
80
0
1
2
Time Period


DRO-1
Wealth
120
100
80
0
1
2
Time Periods

Proposition

Given the sets $\widetilde{\mathcal{U}}_1, \ldots, \widetilde{\mathcal{U}}_T(\mathbf{d}_{T-1})$, the robust counterpart of constraint $\mathbb{E}\left[\sum_{t=1}^{T} h_t(\mathbf{x}_t, \boldsymbol{\xi}_t)\right] \leq B$ is

$$\sup_{P_1 \in \widetilde{\mathcal{U}}_1} \mathbb{E}_{P_1}\Big[h_1(\mathbf{x}_1, \mathbf{d}_1) + \sup_{P_{2|1} \in \widetilde{\mathcal{U}}_2(\mathbf{d}_1)} \Big\{\mathbb{E}_{P_{2|1}}\Big[h_2(\mathbf{x}_2, \mathbf{d}_2) + \cdots + \sup_{P_{T|T-1} \in \widetilde{\mathcal{U}}_T(\mathbf{d}_{T-1})} \Big\{\mathbb{E}_{P_{T|T-1}}\Big[h_T(\mathbf{x}_T, \mathbf{d}_T)\Big]\Big\}\Big]\Big\}\Big] \leq B.$$