Optimization under Connected Uncertainty

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October 23, 2017

Connected Uncertainty Sets

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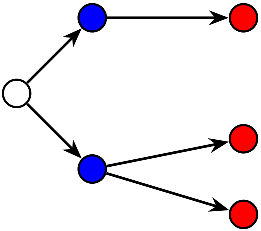
- Current uncertainty realization $\xi_1 \rightarrow$ future realizations ξ_2.



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Example : Ellipsoid Uncertainty set can be expressed as combination of two uncertainty sets.

$$\begin{aligned}\mathcal{U} &= \{\boldsymbol{\xi} = (\xi_1, \xi_2) \mid \|\boldsymbol{\xi}\|_2 \leq \rho\} \\ &= \{\boldsymbol{\xi} = (\xi_1, \xi_2) \mid \xi_1 \in \mathcal{U}_1, \xi_2 \in \mathcal{U}_2(\xi_1)\}\end{aligned}$$

where

$$\mathcal{U}_1 = \left\{\xi_1 \mid |\xi_1| \leq \rho\right\} \qquad \mathcal{U}_2(\xi_1) = \left\{\xi_2 \mid |\xi_2| \leq \sqrt{\rho^2 - \xi_1^2}\right\}$$

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The parameters of the uncertainty set at each period are a function of past realizations.

Past Work

- **Scenario Tress**:Infanger and Morton [1996], De Queiroz and Morton [2013]
- **RO**: Zhao and Zeng [2012], Jiang et al. [2012], Bertsimas and Vayanos [2015], Lorca and Sun [2015,2017], and Nohadani and Roy [2017].
- **DRO**: Analui and Pflug [2014], Xin and Goldberg [2015]

RO Example

Consider

$$\begin{aligned} \max_{\mathbf{x}_1,\mathbf{x}_2} \ & \mathbf{c}_1^\top \mathbf{x}_1 + \mathbf{c}_2^\top \mathbf{x}_2 \\ \text{s.t. } & \mathbf{d}_1^\top \mathbf{x}_1 + \mathbf{d}_2^\top \mathbf{x}_2 \le B \quad \forall \mathbf{d}_2 \in \mathcal{U}_2(\mathbf{d}_1) \quad \forall \mathbf{d}_1 \in \mathcal{U}_1 \\ & \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X} \end{aligned}$$

Uncertainty for $\mathbf{d}_2$ explicitly depends on $\mathbf{d}_1$.

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- Ellipsoid

$$\mathcal{U}_2(\mathbf{d}_1) = \{\mathbf{d}_2 \mid \mathbf{d}_2 = \boldsymbol{\mu}_2(\mathbf{d}_1) + \mathbf{L}_2\mathbf{u}_2 : \|\mathbf{u}_2\|_2 \le r_2\}, \quad \text{(E)}$$

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- Center

$$\boldsymbol{\mu}_2(\mathbf{d}_1) = \mathbf{A}_2\boldsymbol{\mu}_1(\mathbf{d}_0) + \mathbf{F}_2\mathbf{d}_1 + \mathbf{c}_2,$$

DRO Example

$$\sup_{P_1 \in \widetilde{\mathcal{U}}_1} \mathbb{E}_{\mathbf{d_1} \sim P_1} \left[\mathbf{d}_1^\top \mathbf{x}_1 \right] \leq B. \tag{DRO}$$

DRO Example

$$\sup_{P_1 \in \widetilde{\mathcal{U}}_1} \mathbb{E}_{\mathbf{d}_1 \sim P_1} \left[\mathbf{d}_1^\top \mathbf{x}_1 \right] \leq B. \tag{DRO}$$

$$\widetilde{\mathcal{U}}_1 = \Big\{ P_1 \in \mathcal{M} \Big| P_1(\mathbf{d}_1 \in \Xi_1) = 1,\ \underline{\boldsymbol{\mu}}_1 \leq \mathbb{E}_{P_1}[\mathbf{d}_1] \leq \overline{\boldsymbol{\mu}}_1,$$
$$\mathbb{E}_{P_1}[(\mathbf{d}_1 - \boldsymbol{\mu}_1)(\mathbf{d}_1 - \boldsymbol{\mu}_1)^\top] \preceq \boldsymbol{\Sigma}_1 \Big\},$$

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$$\widetilde{\mathcal{U}}_2(\mathbf{d}_1) = \Big\{ P_{2|1} \in \mathcal{M} \Big| P_{2|1}(\mathbf{d}_t \in \Xi_2) = 1,\ \underline{\boldsymbol{\mu}}_2(\mathbf{d}_1) \leq \mathbb{E}_{P_{2|1}}[\mathbf{d}_2] \leq \overline{\boldsymbol{\mu}}_2(\mathbf{d}_1), \mathbb{E}_{P_{2|1}}[(\mathbf{d}_2 - \boldsymbol{\mu}_2(\mathbf{d}_1))(\mathbf{d}_2 - \boldsymbol{\mu}_2(\mathbf{d}_1))^\top] \preceq \boldsymbol{\Sigma}_2(\mathbf{d}_1) \Big\}$$

DRO Example

$$\sup_{P_1 \in \widetilde{\mathcal{U}}_1} \mathbb{E}_{\mathbf{d}_1 \sim P_1} \left[\mathbf{d}_1^\top \mathbf{x}_1 + \sup_{P_2 \in \widetilde{\mathcal{U}}_2(\mathbf{d}_1)} \mathbb{E}_{\mathbf{d}_2 \sim P_2} \left[\mathbf{d}_2^\top \mathbf{x}_2 \right] \right] \leq B. \quad \text{(DRO)}$$

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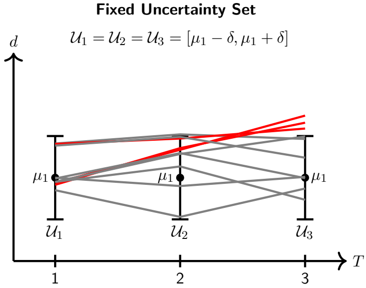
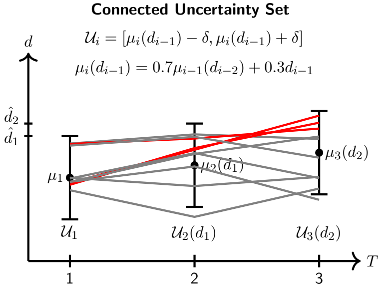
Why do this?

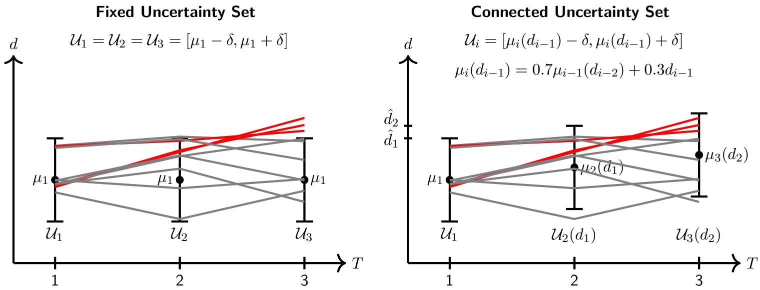


Fixed Uncertainty Set
$\mathcal{U}_1 = \mathcal{U}_2 = \mathcal{U}_3 = [\mu_1 - \delta, \mu_1 + \delta]$
d
μ_1
μ_1
μ_1
$\mathcal{U}_1$
$\mathcal{U}_2$
$\mathcal{U}_3$
T
1
2
3



Connected Uncertainty Set
$\mathcal{U}_i = [\mu_i(d_{i-1}) - \delta, \mu_i(d_{i-1}) + \delta]$
$\mu_i(d_{i-1}) = 0.7\mu_{i-1}(d_{i-2}) + 0.3d_{i-1}$
d
$\hat{d}_2$
$\hat{d}_1$
μ_1
$\mu_2(d_1)$
$\mu_3(d_2)$
$\mathcal{U}_1$
$\mathcal{U}_2(d_1)$
$\mathcal{U}_3(d_2)$
T
1
2
3

Why do this?



- Provides better coverage of uncertainty realizations

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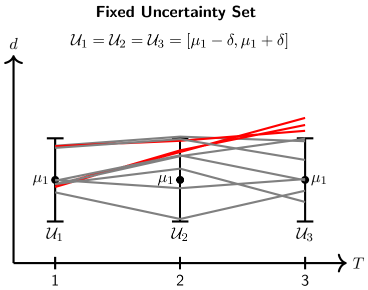

- Provides better coverage of uncertainty realizations
- Incorporates information about structure of uncertainty realizations.

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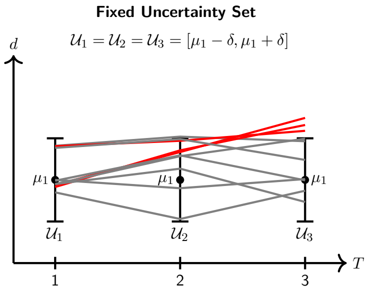

- Provides better coverage of uncertainty realizations
- Incorporates information about structure of uncertainty realizations.
- Provides framework for updating uncertainty sets if desired.

RO: Insights

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$$\sum_{t=1}^{T} \mathbf{d}_t^\top \mathbf{x}_t \leq B$$

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- Linear constraints with polyhedral or ellipsoidal uncertainty sets are reformulated as LPs and SOCPs.

DRO: Insights

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- DRO problems with moment CU sets are reformulated as infinite dimensional problems.
- Infinite dimensional problem are conservatively approximated by limiting number of variables.

Portfolio Optimization

- Two period portfolio optimization problem

$$
\begin{aligned}
\max_{x_1, x_2} \ \min_{P \in \mathcal{U}} \ & \mathbb{E}[u(r_1^\top x_1) + u(r_2^\top x_2)] \\
\text{s.t.} \ & \sum_{i=1}^{n} x_{1i} = 1 \\
& \sum_{i=1}^{n} x_{2i} = 1 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

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- Maximize $u(z) = \min_k a_k z + b_k$

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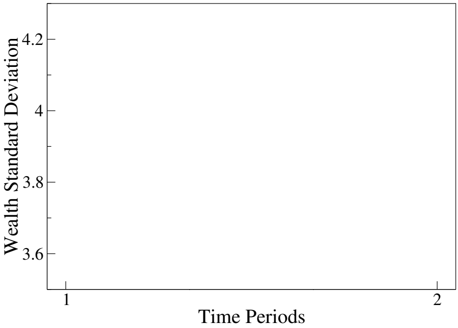
- Moment based uncertainty set. $\boldsymbol{\mu}_2(\mathbf{d}_1) = \boldsymbol{\mu}_0 + \mathbf{A}\boldsymbol{\mu}_1 + \mathbf{B}\mathbf{d}_1$
- Estimate performance of solutions on actual return data.

DRO-1: $\mu_2 = \mu_1$,
DRO-2: $\mu_2 = \mu_0 + A\mu_1 + B\mu_1$



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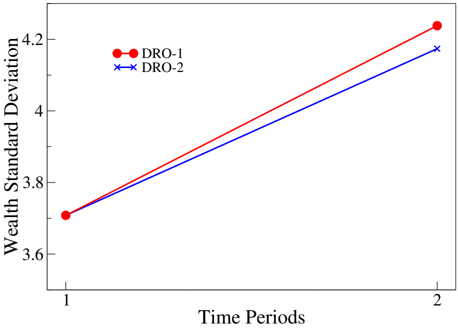

Portfolio Optimization

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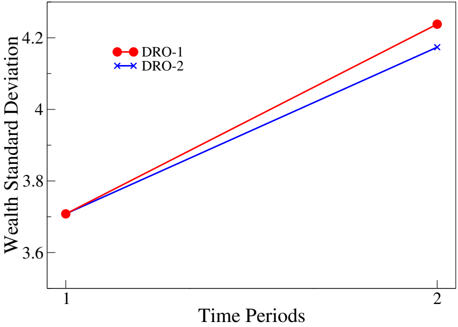

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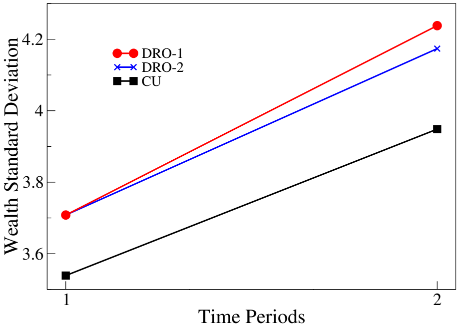

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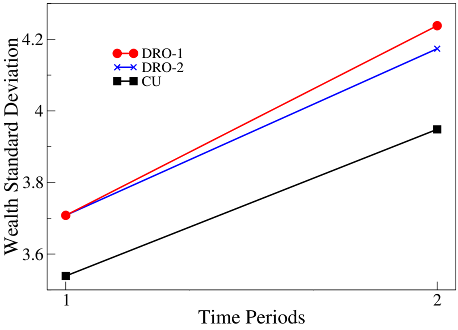

- Lower uncertainty in wealth for almost the same average returns.

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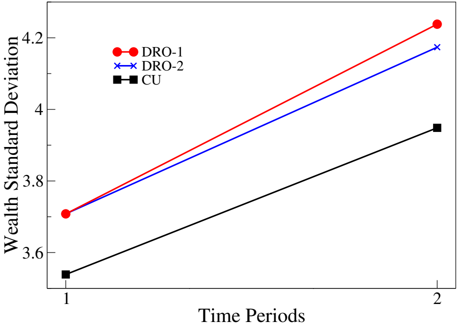

- Lower uncertainty in wealth for almost the same average returns.
- More difficult to solve.

Robust Knapsack

- Two period knapsack problem with uncertain weight coefficients.

$$\begin{aligned}\min_{\mathbf{x}_1,\mathbf{x}_2}\ & \mathbf{c}_1^\top \mathbf{x}_1 + \mathbf{c}_2^\top \mathbf{x}_2 \\ \text{s.t. } & \mathbf{d}_1^\top \mathbf{x}_1 + \mathbf{d}_2^\top \mathbf{x}_2 \le B \ \ \forall \mathbf{d}_2 \in \mathcal{U}_2(\mathbf{d}_1) \ \ \forall \mathbf{d}_1 \in \mathcal{U}_1 \\ & \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}\end{aligned}$$

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- Ellipsoidal uncertainty sets. Parameters from samples.

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- Ellipsoidal uncertainty sets. Parameters from samples.
- Solution quality evaluated on new uncertainty samples.

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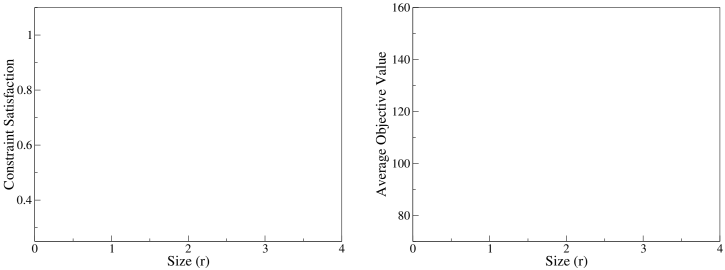
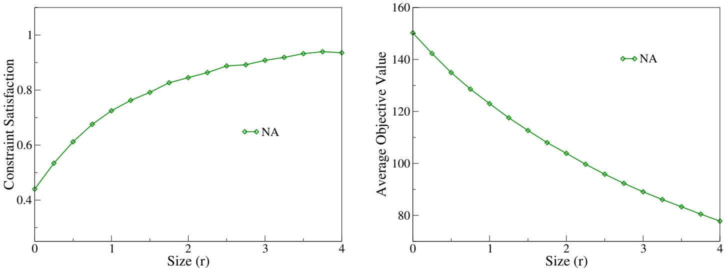
- Ellipsoidal uncertainty sets. Parameters from samples.
- Solution quality evaluated on new uncertainty samples.
- Performance evaluated with Normal and Lognormal distributions.

Robust Knapsack



- Better constraint satisfaction at the price of lower objective value.

Robust Knapsack



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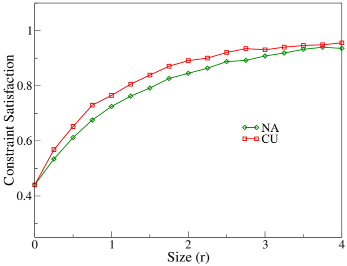
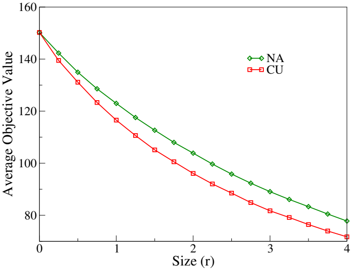
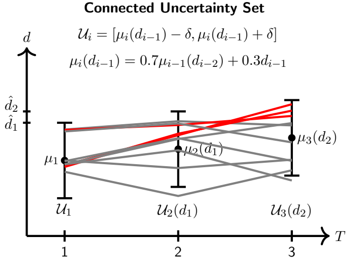




- Better constraint satisfaction at the price of lower objective value.

Conclusions

- In many problems, uncertainty depends on past realizations.
- Connected uncertainty sets incorporate this behavior and lead to less variable solutions.
- The reformulated problems are difficult → decision rules and conservative approximations.



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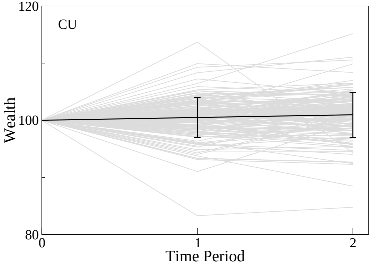
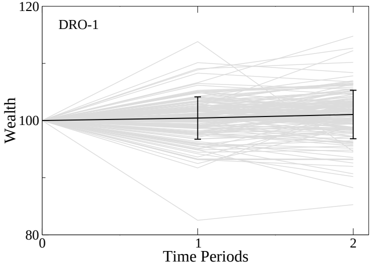

CU
Wealth
120
100
80
0
1
2
Time Period


DRO-1
Wealth
120
100
80
0
1
2
Time Periods

Proposition

Given the sets $\widetilde{\mathcal{U}}_1, \ldots, \widetilde{\mathcal{U}}_T(\mathbf{d}_{T-1})$, the robust counterpart of constraint $\mathbb{E}\left[\sum_{t=1}^{T} h_t(\mathbf{x}_t, \boldsymbol{\xi}_t)\right] \leq B$ is

$$\sup_{P_1 \in \widetilde{\mathcal{U}}_1} \mathbb{E}_{P_1}\Big[h_1(\mathbf{x}_1, \mathbf{d}_1) + \sup_{P_{2|1} \in \widetilde{\mathcal{U}}_2(\mathbf{d}_1)} \Big\{\mathbb{E}_{P_{2|1}}\Big[h_2(\mathbf{x}_2, \mathbf{d}_2) + \cdots + \sup_{P_{T|T-1} \in \widetilde{\mathcal{U}}_T(\mathbf{d}_{T-1})} \Big\{\mathbb{E}_{P_{T|T-1}}\Big[h_T(\mathbf{x}_T, \mathbf{d}_T)\Big]\Big\}\Big]\Big\}\Big] \leq B.$$