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Dynamic Capacity Management for Deferred Surgeries

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Health Security

- What is health security?

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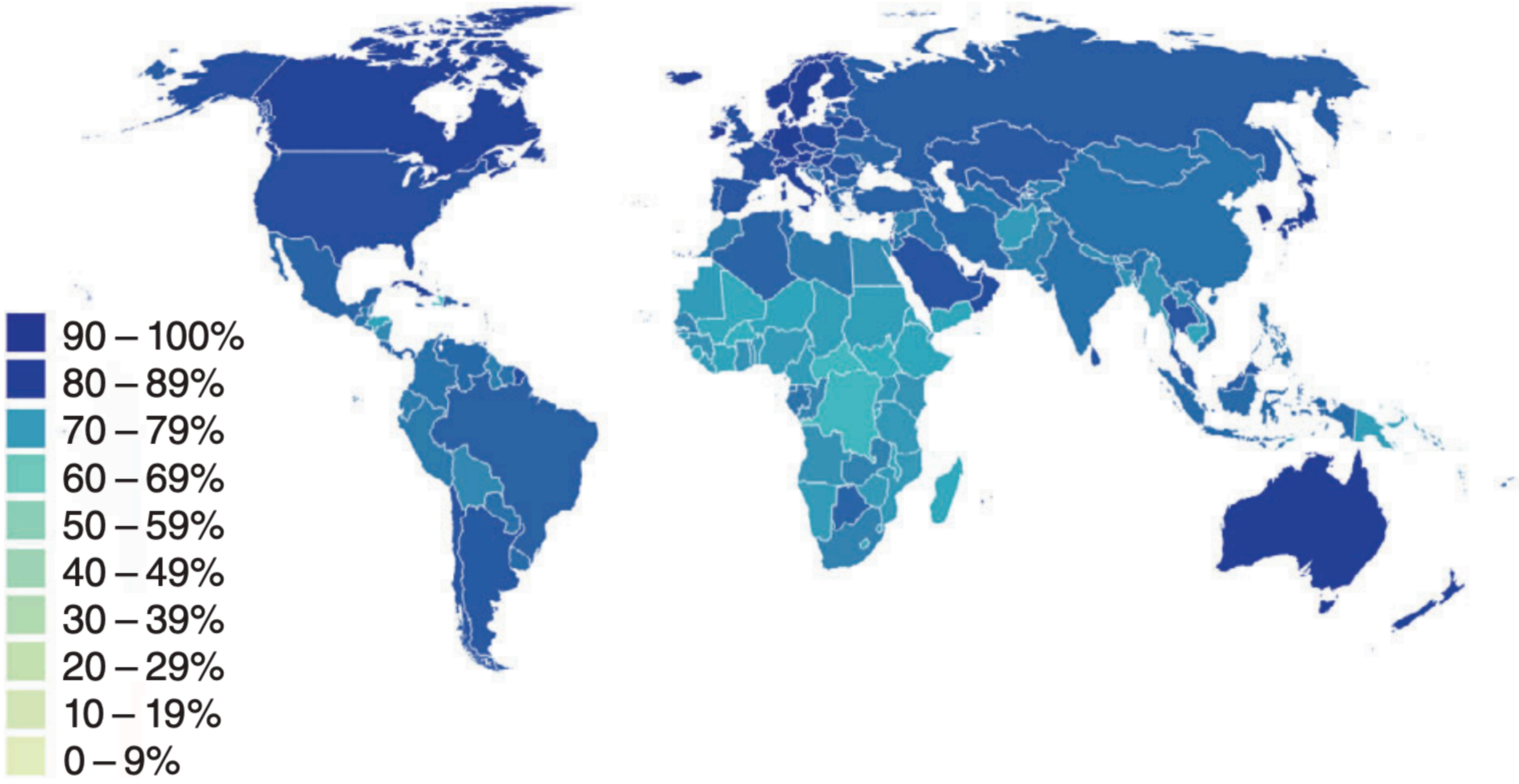
- Existing efforts
  - Focus on preparedness
  - Emphasis on detection and spread prevention
- Our Goal
  - Address resilience
  - Emphasis on responsiveness and adaptability of hospitals

Deferred Elective Surgeries

12-week cancellation rates of surgery for benign disease (March to May 2020)



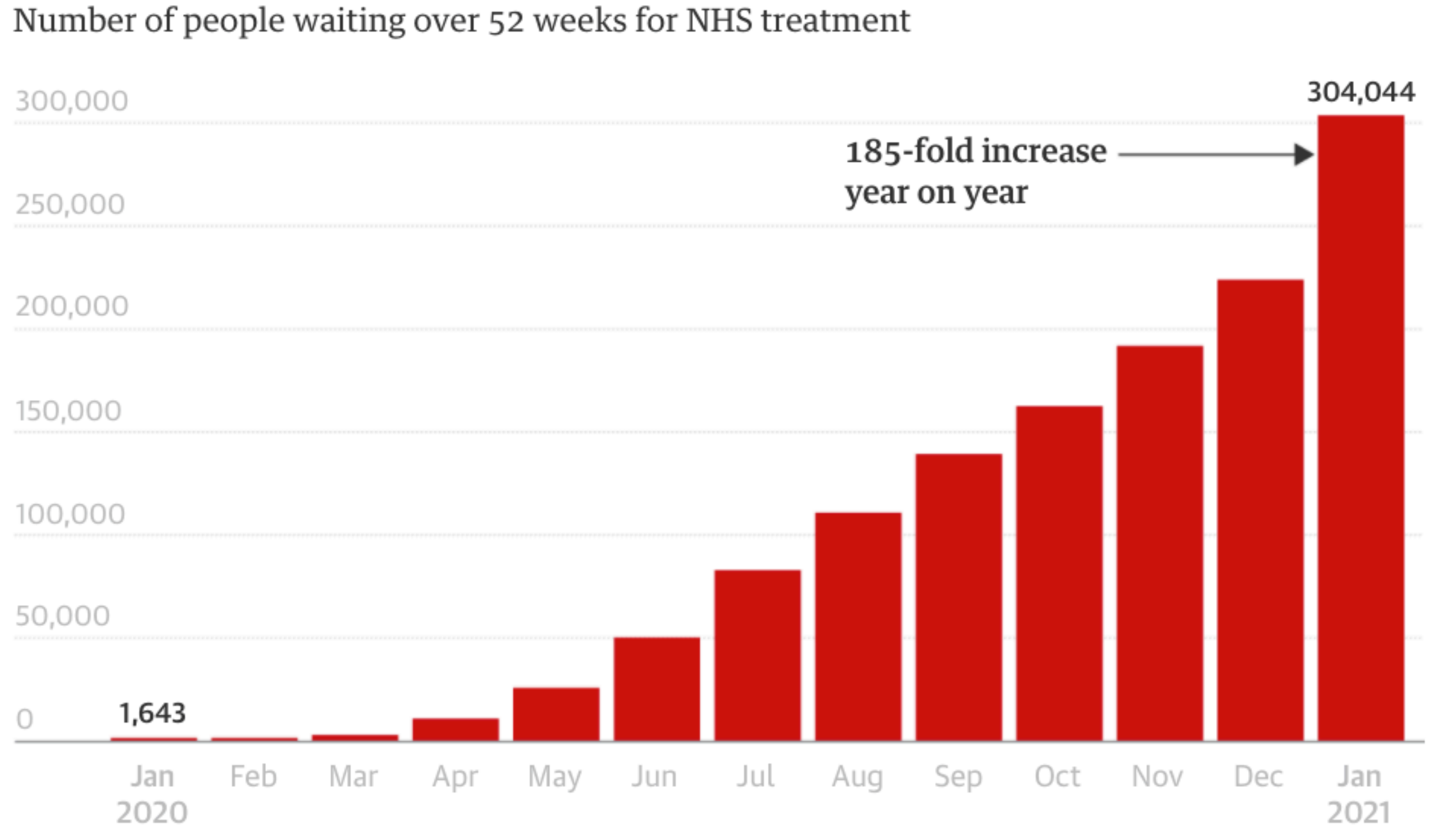
Source: COVIDSurg Collaborative (2020) Elective surgery cancellations due to the COVID-19 pandemic: global predictive modeling to inform surgical recovery plans. British Journal of Surgery, 107(11): 1440-1449.

The Guardian

New Covid wave could worsen NHS surgery backlog, experts warn

Relaxation of rules and sharp rise in B.1.617.2 variant cause concern, as millions wait for hospital treatment

There has been a huge increase in the number of people waiting more than a year for NHS care since the start of the Covid pandemic



Source: D. Campbell. ‘A truly frightening backlog’: ex-NHS chief warns of delays in vital care. The Guardian, April 2, 2021 / N. Davis and D. Campbell. New Covid wave could worsen NHS surgery backlog, experts warn. The Guardian, May 20, 2021.

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Cost of Deferred Elective Surgeries

- Increased (financial and social) costs, due to more costly treatment for more advanced diseases.

- Significant financial loss for hospitals
  - Average monthly loss of revenue of the U.S. hospitals is $50.7 billion for March-June 2020 (Meredith et al. 2020).
  - Elective surgeries account for 43% of gross revenue of the U.S. hospitals (Tonna et al. 2020).

Source: Meredith, High, and Freischlag (2020) Preserving elective surgeries in the COVID-19 pandemic and the future. JAMA 324(17):1725-1726. Tonna, Hanson, Cohan, McCrum, Horns, Brooke, Das, Kelly, Campbell, and Hotaling (2020) Balancing revenue generation with capacity generation: case distribution, financial impact and hospital capacity changes from cancelling or resuming elective surgeries in the US during COVID-19. BMC Health Services Research 20(1):1-7.

Capacity Management for Deferred Surgeries

- Expanding surgical capacity of hospitals is *necessary*.
  - Quote from Jain et al. (2020) on elective orthopedic surgery in the U.S.:

> *"When the healthcare system recovers to the pre-pandemic forecasted full capacity, there will be a cumulative backlog of >1 million surgical cases at 2 years after the end of deferment. ... it appears to be impossible to close the gap on the accumulative backlog."*

Source: Jain, Jain, and Aggarwal (2020) SARS-Cov-2 impact on elective orthopedic surgery: implications for post-pandemic recovery. The Journal of Bone and Joint Surgery. Ljungqvist, Nelson, and Demartines (2020) The post COVID-19 surgical backlog: Now is the time to implement enhanced recovery after surgery. World Journal of Surgery 44(10):3197-3198. Salenger et al. (2020) The surge after the surge: cardiac surgery post-COVID-19. The Annals of the Thoracic Surgery.

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  - No expansion, or expanding capacities by pre-determined amounts.
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➡ **Goal**: Develop an *optimization-based methodology* to dynamically manage surgical capacity for deferred surgeries, while maximizing the profit with service requirements.

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Problem Set-up

- $\mathbf{C}_B = (C_{B,1}, \cdots, C_{B,t})$: base expansion decision.
- $\mathbf{C} = (C_1, \cdots, C_t)$: expedite expansion decision.
- $\mathbf{u}_t = (u_t^{(-L)}, \cdots, u_t^{(t)})$: number of deferred surgeries.
  - $u_t^{(\tau)}$ is the number of deferred surgeries initially scheduled at τ but not performed until t.
  - $u_t^{(t)} = d_t$ is the uncertain demand at t.
- $\mathbf{x}_t = (x_t^{(-L)}, \cdots, x_t^{(t)})$: surgery decision.
- $\mathbf{w}_t = (w_t^{(-L)}, \cdots, w_t^{(t)})$: uncertain number of departing patients.
- State dynamics and constraints:

$$\left\{ \begin{array}{l} u_{t+1}^{(\tau)} = u_t^{(\tau)} - x_t^{(\tau)} - w_t^{(\tau)} \quad \forall \tau = -L, \cdots, t, \ \forall t = 1, \cdots, T \\ x_t^{(\tau)} \leq u_t^{(\tau)}, \ \sum_{\tau=-L}^{t} x_t^{(\tau)} \leq \hat{C}_t + C_{B,t} + C_t, \ (\mathbf{C}_B, \mathbf{C}) \in \mathcal{C} \end{array} \right\}$$

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State dynamics

Demand & capacity constraints

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Expansion constraints

Dynamic Programming Formulation

- Cost at time t :

$$
\begin{aligned}
H_t(C_{B,t}, C_t, \mathbf{u}_t, \mathbf{x}_t, \mathbf{w}_t) := \ & b_{B,t}(\hat{C}_t + C_{B,t}) + b_t C_t \\
& + c_t \sum_{\tau=-L}^{t} x_t^{(\tau)} + \sum_{\tau=-L}^{t} p_{t-\tau}(u_t^{(\tau)} - x_t^{(\tau)} - w_t^{(\tau)}) + \sum_{\tau=-L}^{t} f_{t-\tau} w_t^{(\tau)}
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Dynamic Programming Formulation

- Cost at time t :

$b_{B,t}$: Base expansion cost

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$p_{t-\tau}$: Defer cost

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c_t : Surgery cost $p_{t-\tau}$: Defer cost $f_{t-\tau}$: Departure cost

- Dynamic programming (DP) problem:

$$\min_{\mathbf{C}_B, C_1} \mathbb{E}_{d_1}\left[\min_{\mathbf{x}_1} \mathbb{E}_{\mathbf{w}_1}\left[H_1(\cdot) + \min_{C_2} \mathbb{E}_{d_2}\left[\min_{\mathbf{x}_2} \mathbb{E}_{\mathbf{w}_2}\left[H_2(\cdot) + \cdots + \min_{C_T} \mathbb{E}_{d_T}\left[\min_{\mathbf{x}_T} \mathbb{E}_{\mathbf{w}_T}\left[H_T(\cdot)\right]\right] \cdots\right]\right.\right.\right.$$

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- Solving this problem as a stochastic DP may not be appropriate.
  - Uncertain departure $\mathbf{w}_t$ should be described *endogenously*; it should depend on $\mathbf{u}_t$ and $\mathbf{x}_t$.
    - This causes *multilinear uncertainty*, which makes the problem very challenging to solve.
  - *Limited distributional information* for uncertain parameters: Only a few data points are available with large variability; poor approximation can significantly undermine the performance.

Demand and Departure Uncertainty

- Departing patients $\mathbf{w}_t$ depends on $\mathbf{u}_t$ and $\mathbf{x}_t$:

$$\mathbf{w}_t \leq \mathbf{u}_t - \mathbf{x}_t \text{ almost surely.}$$

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- Now $\mathbf{u}_t$ is described via **multilinear** functions of θ_t , d_t , and $\mathbf{x}_t$ as

$$u_t^{(\tau)} = \left(\prod_{k=\max(\tau,1)}^{t-1} \theta_k \right) d_\tau - \sum_{t'=\max(\tau,1)}^{t-1} \left(\prod_{k=t'}^{t-1} \theta_k \right) x_{t'}^{(\tau)} \qquad \forall \tau = -L, \cdots, t \ \forall t \in [T] \, .$$

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- We take a (distributionally) robust optimization approach to address this multilinearity.

Overview

1. Introduction
2. Robust Optimization (RO) Approach
   - Tree of Uncertainty Products
   - Decision Rule Approximations
3. Distributionally Robust Optimization (DRO) Approach
   - Mean-Mean Absolute Deviation (MAD) Ambiguity Sets
   - Sample Average Approximations
4. A Case Study for Hernia Surgeries
   - Performance Improvement
   - Structural Insights
   - Sensitivity Analysis
5. Conclusions

Robust Optimization

- Methodology to tackle optimization problems under uncertainty
- Assumes that the uncertain parameter (denoted by ξ) lies inside a set (denoted by $\mathcal{U}$)
- **Objective**: $f(x, \xi)$ and **Constraint**: $g(x, \xi) \leq 0$, then

$$\min_{x} \max_{\xi \in \mathcal{U}} f(x, \xi)$$
$$\text{s.t. } g(x, \xi) \leq 0 \quad \forall \xi \in \mathcal{U}$$

- Minimizes the worst case of the objective
- Constraints need to be satisfied for every realization
- We use a multistage variant
    - Solution at stage t depends on uncertainty at stage $t-1$

Formulation

$$\min_{C_t(\cdot),\mathbf{x}_t(\cdot)} \max_{\theta_{[T]}\in\Theta, d_{[T]}\in\mathcal{U}} \sum_{t\in[T]} G_t\left(C_t(\theta_{[t-1]}, d_{[t-1]}), \mathbf{x}_{[t]}(\theta_{[t-1]}, d_{[t]}), \theta_{[t]}, d_{[t]}\right)$$

$$\text{s.t.} \sum_{t'=\max(\tau,1)}^{t} \left(\prod_{k=t'}^{t-1}\theta_k\right) x_{t'}^{(\tau)}(\theta_{[t'-1]}, d_{[t']}) \le \left(\prod_{k=\max(\tau,1)}^{t-1}\theta_k\right) d_\tau \quad \forall\theta_{[T]}\in\Theta, d_{[T]}\in\mathcal{U}, \tau\in[-L:t], t\in[T]$$

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$$\mathbf{x}_t(\theta_{[t-1]}, d_{[t]}) \in \mathbb{R}_+^{t+L} \quad \forall\theta_{[T]}\in\Theta, d_{[T]}\in\mathcal{U}, t\in[T]$$

$$(\mathbf{C}_B, C_1, C_2(\theta_1, d_1), \cdots, C_T(\theta_{[T-1]}, d_{[T-1]})) \in \mathcal{C} \quad \forall\theta_{[T]}\in\Theta, d_{[T]}\in\mathcal{U},$$

where $G_t(C_t, \mathbf{x}_{[t]}, \theta_{[t]}, d_{[t]}) :=$

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$$+ \sum_{\tau=-L}^{t} (p_{t-\tau} - f_{t-\tau})\left[\left(\prod_{k=\max(\tau,1)}^{t}\theta_k\right) d_\tau - \sum_{t'=\max(\tau,1)}^{t}\left(\prod_{k=t'}^{t}\theta_k\right) x_{t'}^{(\tau)}\right].$$

Formulation

Can be $\mathbb{E}$ (stochastic), or $\sup \mathbb{E}$ (distributionally robust)

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Formulation

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Multilinear uncertainty

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Multilinear Uncertainty

- $\boldsymbol{\xi} := (\theta_1, \cdots, \theta_T, d_1, \cdots, d_T)$ with the uncertainty set $\Xi := \Theta \times \mathscr{U}$.
- Objective functions and constraints are given as multilinear robust constraints:

$$\mathbf{p}^\top \boldsymbol{\xi} + \sum_{n=1}^{N} q_n \cdot g_n(\boldsymbol{\xi}) \geq q_0 \quad \forall \boldsymbol{\xi} \in \Xi, \text{ where } g_n(\boldsymbol{\xi}) := \prod_{i \in \mathscr{I}_n} \xi_i.$$

Example:

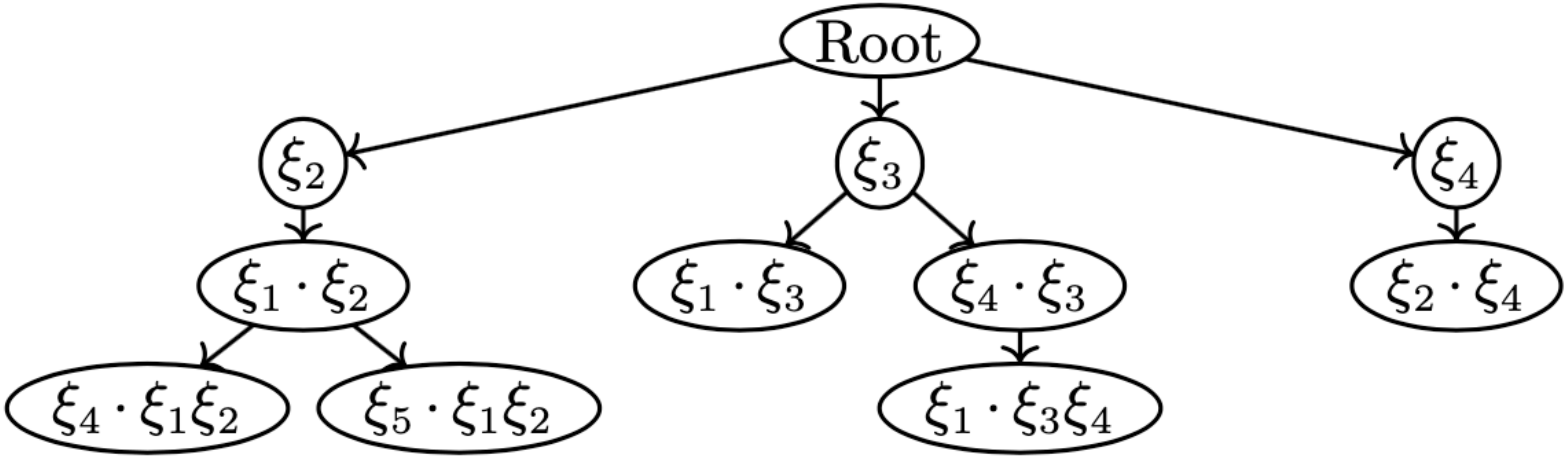
$$\mathbf{p}^\top \boldsymbol{\xi} + q_1\xi_1\xi_2 + q_2\xi_2\xi_3 + q_3\xi_1\xi_2\xi_3 \geq q_0, \text{ where}$$
$$\mathscr{I}_1 = \{1,2\}, \mathscr{I}_2 = \{2,3\}, \mathscr{I}_3 = \{1,2,3\}.$$

- "Typical" robust constraints are linear, i.e., $q_n = 0 \; \forall n \geq 1$.
- Multilinear robust constraints are generally not tractable.

Tree of Uncertainty Products: Example

$\mathbf{z}^{(0)} = (0,0,0,0,0)$

Root node (Node 0)



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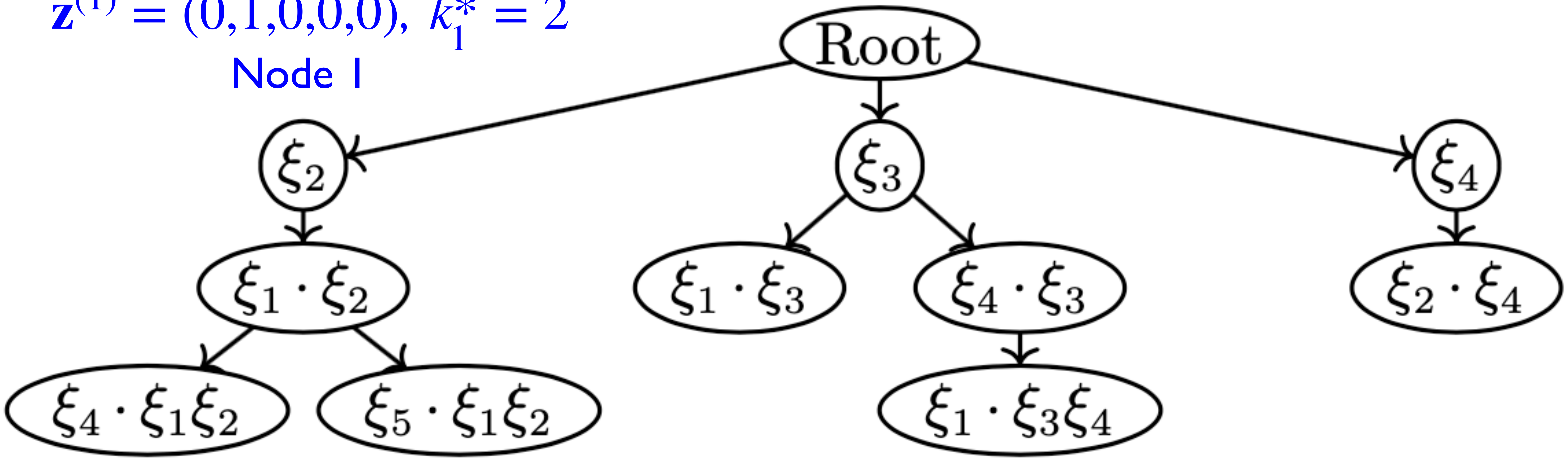
$\mathbf{z}^{(1)} = (0,1,0,0,0),\ k_1^* = 2$

Node 1



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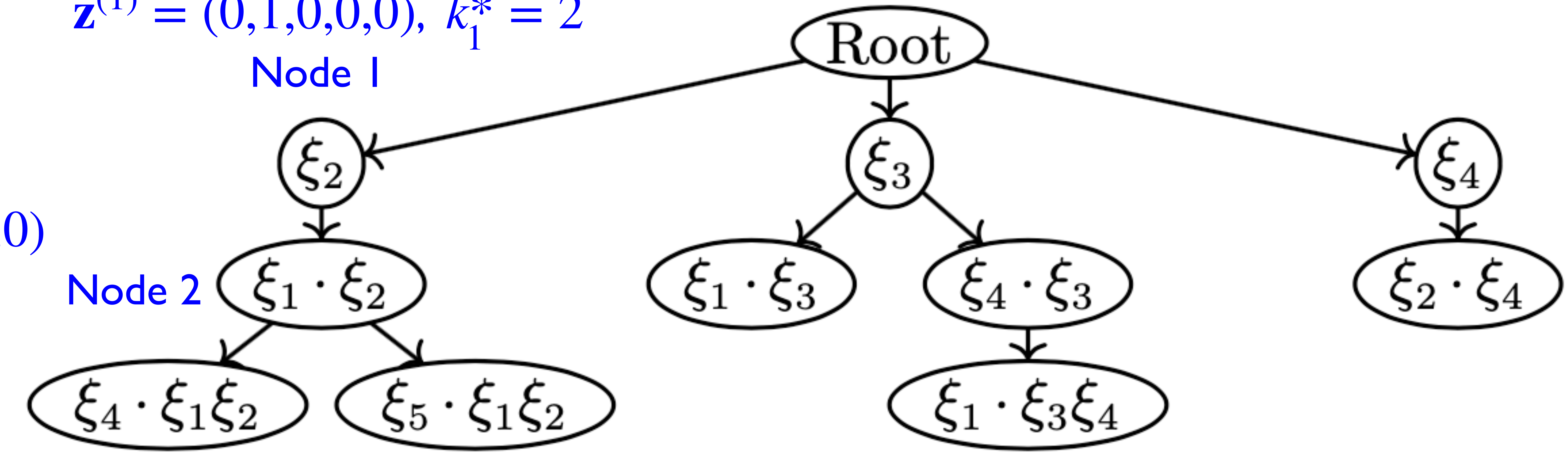
$k_2^* = 1$

Node 2



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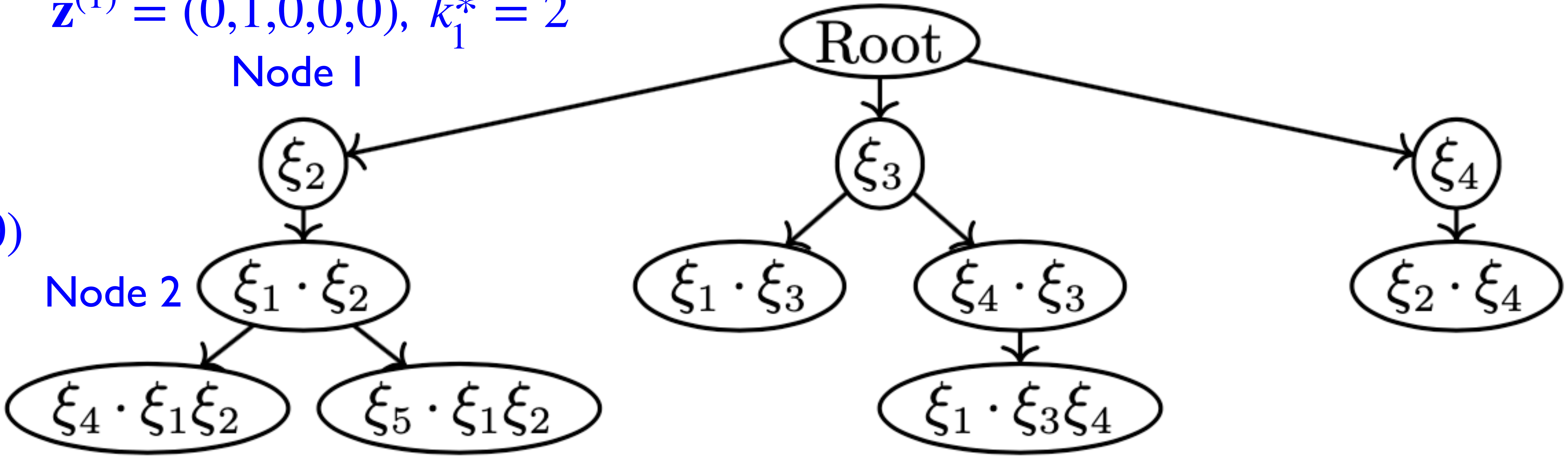

Node 3

$\mathbf{z}^{(3)} = (1,1,0,1,0)$

$k_3^* = 4$

Tree of Uncertainty Products: Example

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$\mathbf{z}^{(1)} = (0,1,0,0,0),\ k_1^* = 2$

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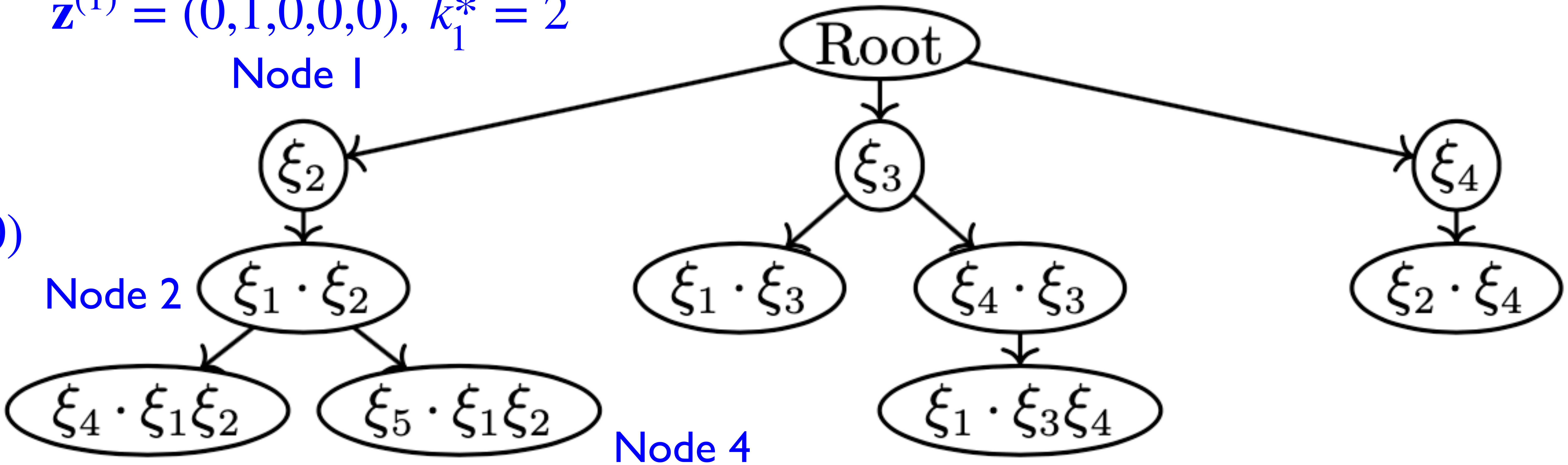

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$\mathbf{z}^{(3)} = (1,1,0,1,0)$

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Node 4

$\mathbf{z}^{(4)} = (1,1,0,0,1)$

$k_4^* = 5$

Lifting with Tree of Uncertainty Products

- $\boldsymbol{\xi} := (\theta_1, \cdots, \theta_T, d_1, \cdots, d_T)$ with the uncertainty set $\Xi := \Theta \times \mathcal{U}$.
- Lifted uncertainty set by using recursive binary McCormick relaxation between node i and its parent node $\ell(i)$

$$\overline{\Xi} := \left\{ (\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbb{R}^{K+N} \;\middle|\; \begin{array}{ll} \boldsymbol{\xi} \in \Xi,\ \eta_i = \xi_{k_i^*} & \forall i : \ell(i) = 0 \\ \eta_i \geq \overline{\eta}_{\ell(i)} \xi_{k_i^*} + \overline{\xi}_{k_i^*} \eta_{\ell(i)} - \overline{\eta}_{\ell(i)} \overline{\xi}_{k_i^*} & \forall i : \ell(i) \neq 0 \\ \eta_i \geq \underline{\eta}_{\ell(i)} \xi_{k_i^*} + \underline{\xi}_{k_i^*} \eta_{\ell(i)} - \underline{\eta}_{\ell(i)} \underline{\xi}_{k_i^*} & \forall i : \ell(i) \neq 0 \\ \eta_i \leq \overline{\eta}_{\ell(i)} \xi_{k_i^*} + \underline{\xi}_{k_i^*} \eta_{\ell(i)} - \overline{\eta}_{\ell(i)} \underline{\xi}_{k_i^*} & \forall i : \ell(i) \neq 0 \\ \eta_i \leq \underline{\eta}_{\ell(i)} \xi_{k_i^*} + \overline{\xi}_{k_i^*} \eta_{\ell(i)} - \underline{\eta}_{\ell(i)} \overline{\xi}_{k_i^*} & \forall i : \ell(i) \neq 0 \end{array} \right\}$$

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McCormick relaxation between

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McCormick relaxation between

- η_i is a lifted uncertain parameter for a node i of the tree of uncertainty products.

Approximating Multilinear Constraints

$$(1) \quad \mathbf{p}^\top \boldsymbol{\xi} + \sum_{n=1}^{N} q_n g(\boldsymbol{\xi}; \mathbf{z}^{(n)}) \geq q_0 \quad \forall \boldsymbol{\xi} \in \Xi$$

Multilinear in $\boldsymbol{\xi}$

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Theorem

If $\Xi = \times_{n=1}^{N} [0, \bar{\xi}_n]$ and the tree of uncertainty product satisfies that $k_i^* \neq k_j^*$ for any $i \neq j, i, j \in \mathcal{N}$, then (1) is equivalent to (2).

Each lifted variable η_n becomes a tight approximate of $g(\boldsymbol{\xi}; \mathbf{z}^{(n)})$.

Extends current literature on convex relaxation of sum of multilinear functions (Ryoo and Sahinidis 2001, Luedtke et al. 2012)

Decision Rule Approximations

- Approximate decision functions by parametric functions

$$x(\xi) \leq 0 \ \ \forall \xi \in \mathcal{U} \text{ by } x_0 + X\xi \leq 0 \forall \xi \in \mathcal{U}$$

- Employ **decision rules**, e.g., Linear decision rules

$$x_t^{(\tau)} = w_t^{(\tau)} + \sum_{t'=1}^{t-1} W_{t,t'}^{(\tau)} \theta_{t'} + \sum_{t'=1}^{t} \hat{W}_{t,t'}^{(\tau)} d_{t'}, C_t = v_t + \sum_{t'=1}^{t-1} V_{t,t'} \theta_{t'} + \sum_{t'=1}^{t-1} \hat{V}_{t,t'} d_{t'}.$$

- Form a tree of uncertainty products and approximate with lifted uncertainty sets.

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Generalizable to *multilinear* decision rules!

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Distributionally Robust Optimization

- Methodology to tackle optimization problems under uncertainty
- Assumes that the uncertain parameter (denoted by ξ) lies has a distribution $\mathbb{P}$ that lies inside an ambiguity set (denoted by $\mathscr{A}$)
- **Objective**: $f(x, \xi)$ and **Constraint**: $g(x, \xi) \leq 0$, then

$$\min_{x} \max_{\mathbb{P} \in \mathscr{A}} \mathbb{E}_{\xi \sim \mathbb{P}}[f(x, \xi)]$$
$$\text{s.t. } g(x, \xi) \leq 0 \quad \forall \xi \in \mathscr{U}$$

- Minimizes the worst-case expected value of the objective
- Constraints need to be satisfied for every realization

Distributionally Robust Optimization

- We use a multistage variant
  - Solution at stage t depends on uncertainty at stage $t-1$

$$\min_{\mathbf{c}_B, C_1} \sup_{F_{d_1}} \mathbb{E}_{d_1}\left[\min_{\mathbf{x}_1} \sup_{F_{\mathbf{w}_1}} \mathbb{E}_{\mathbf{w}_1}\left[H_1(\cdot) + \min_{C_2} \sup_{F_{d_2}} \mathbb{E}_{d_2}\left[\min_{\mathbf{x}_2} \sup_{F_{\mathbf{w}_2}} \mathbb{E}_{\mathbf{w}_2}\left[H_2(\cdot) + \cdots + \min_{C_T} \sup_{F_{d_T}} \mathbb{E}_{d_T}\left[\min_{\mathbf{x}_T} \sup_{F_{\mathbf{w}_T}} \mathbb{E}_{\mathbf{w}_T}\left[H_T(\cdot)\right]\right] \cdots\right]\right]\right]\right]$$

Mean-MAD Ambiguity Sets

Definition

For the set of non-negative Borel measurable functions $\mathscr{M}_+(\mathbb{R}^{2T}), \lambda_{\theta_t}, \lambda_{d_t} \geq 0$,
$0 \leq \underline{\theta}_t < \hat{\theta}_t < \bar{\theta}_t \leq 1$, and $0 \leq \underline{d}_t < \hat{d}_t < \bar{d}_t$, mean-MAD ambiguity set $\mathscr{F}$ is defined as

$$\mathcal{F} = \left\{ F \in \mathcal{M}_+(\mathbb{R}^{2T}) \;\middle|\; \begin{array}{ll} \mathbb{P}_F\left(\theta_t \in \left[\underline{\theta}_t, \bar{\theta}_t\right]\right) = 1,\ \mathbb{E}_F\left[\theta_t\right] = \widehat{\theta}_t,\ \mathbb{E}_F\left[\left|\theta_t - \widehat{\theta}_t\right|\right] \leq \lambda_{\theta_t} & \forall t \in [T] \\ \mathbb{P}_F\left(d_t \in \left[\underline{d}_t, \bar{d}_t\right]\right) = 1,\ \mathbb{E}_F\left[d_t\right] = \widehat{d}_t,\ \mathbb{E}_F\left[\left|d_t - \widehat{d}_t\right|\right] \leq \lambda_{d_t} & \forall t \in [T] \\ \{\theta_{[T]}, d_{[T]}\} \text{ are mutually independent} & \end{array} \right\}.$$

- $\underline{\theta}_t, \bar{\theta}_t, \underline{d}_t, \bar{d}_t$: lower and upper support of θ_t and d_t.
- $\hat{\theta}_t, \hat{d}_t$: expectation of θ_t and d_t.
- $\lambda_{\theta_t}, \lambda_{d_t}$: mean-absolute deviation bound of θ_t and d_t.

All of them can be easily estimated from (small) data!

Reformulation

Theorem

With the mean-MAD ambiguity set $\mathscr{F}$, the multistage DRO problem is reformulated as a stochastic optimization problem with *three-points discrete distributions* for each uncertain parameter.

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- Under mean and MAD constraints, the worst-case probability distribution is *always* fixed, supported over lower and upper bounds, and their means.
- **Insight**: There exists a class of stochastic optimization problems *whose solutions are distributionally robust*!

Sample Average Approximation

- By using scenario trees, we can handle multilinear uncertainty.
- The reformulated problem has finite (3^{2T}) scenarios.
- We use Sample Average Approximation to compute tractable approximate solutions.

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- Our analysis estimates current backlog as 4 months of average (pre-pandemic) monthly demand.
- Three methods are implemented and compared:
  - RO: robust optimization-based method
  - DRO: distributionally robust optimization-based method
  - Det100: temporally increase capacity by at most 100% (for ~5 months)

Performance Improvement

D	Det100			RO			DRO		
	Mean	CVaR75	CVaR90	Mean	CVaR75	CVaR90	Mean	CVaR75	CVaR90
$D=2$	-3631 (0.0)	-2803 (0.0)	-2531 (0.0)	-3648 (0.49)	-2947 (5.14)	-2719 (7.42)	-3820 (5.21)	-2946 (5.09)	-2595 (2.56)
$D=4$	-4741 (0.0)	-4361 (0.0)	-4205 (0.0)	-4770 (0.61)	-4428 (1.52)	-4280 (1.77)	-4903 (3.41)	-4420 (1.34)	-4225 (0.47)

- Both RO and DRO policies achieve better objective values (costs) than deterministic policies.
- DRO performs better in expectation (mean), but RO performs better at higher risk (CVaR90).

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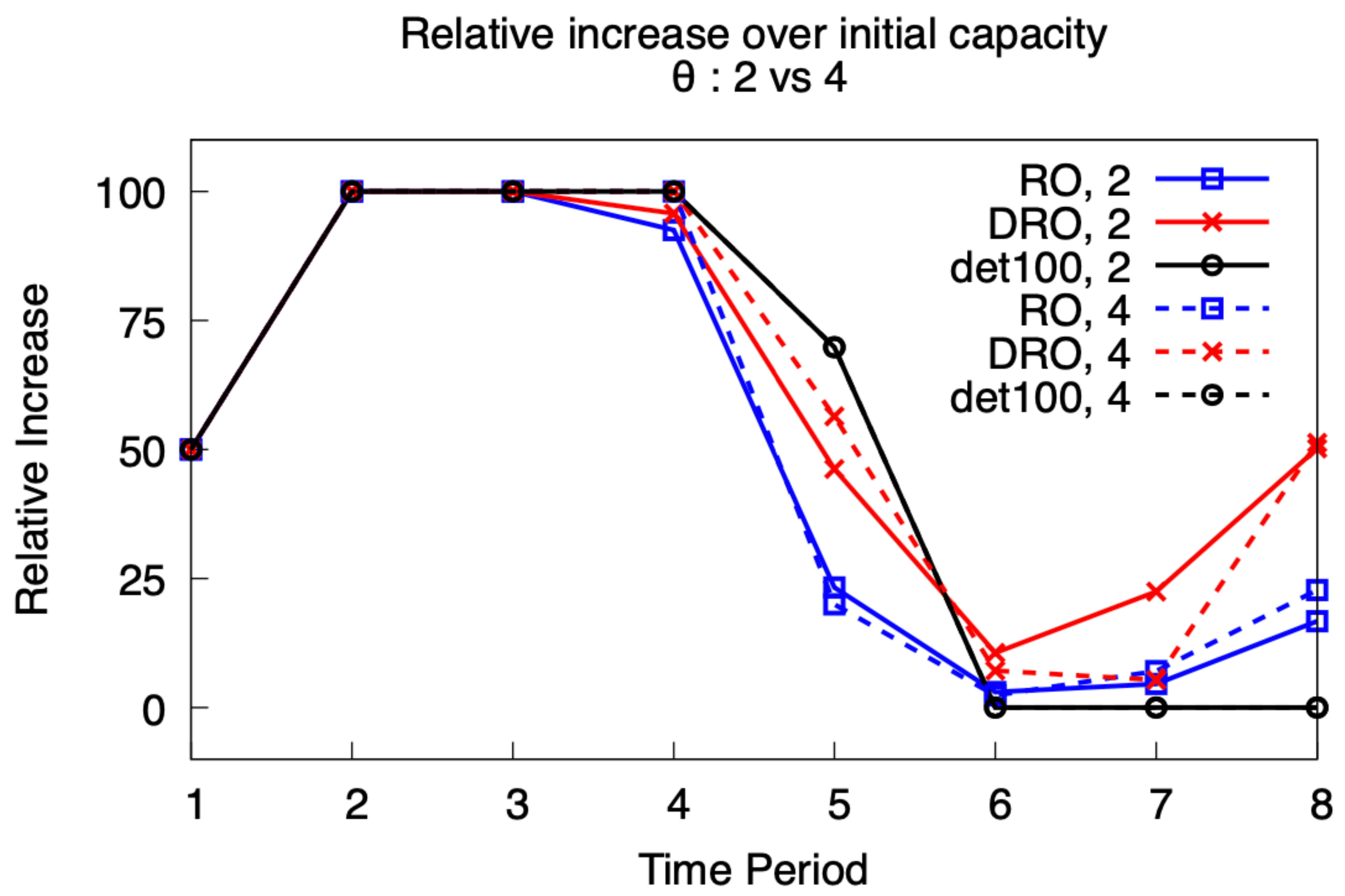
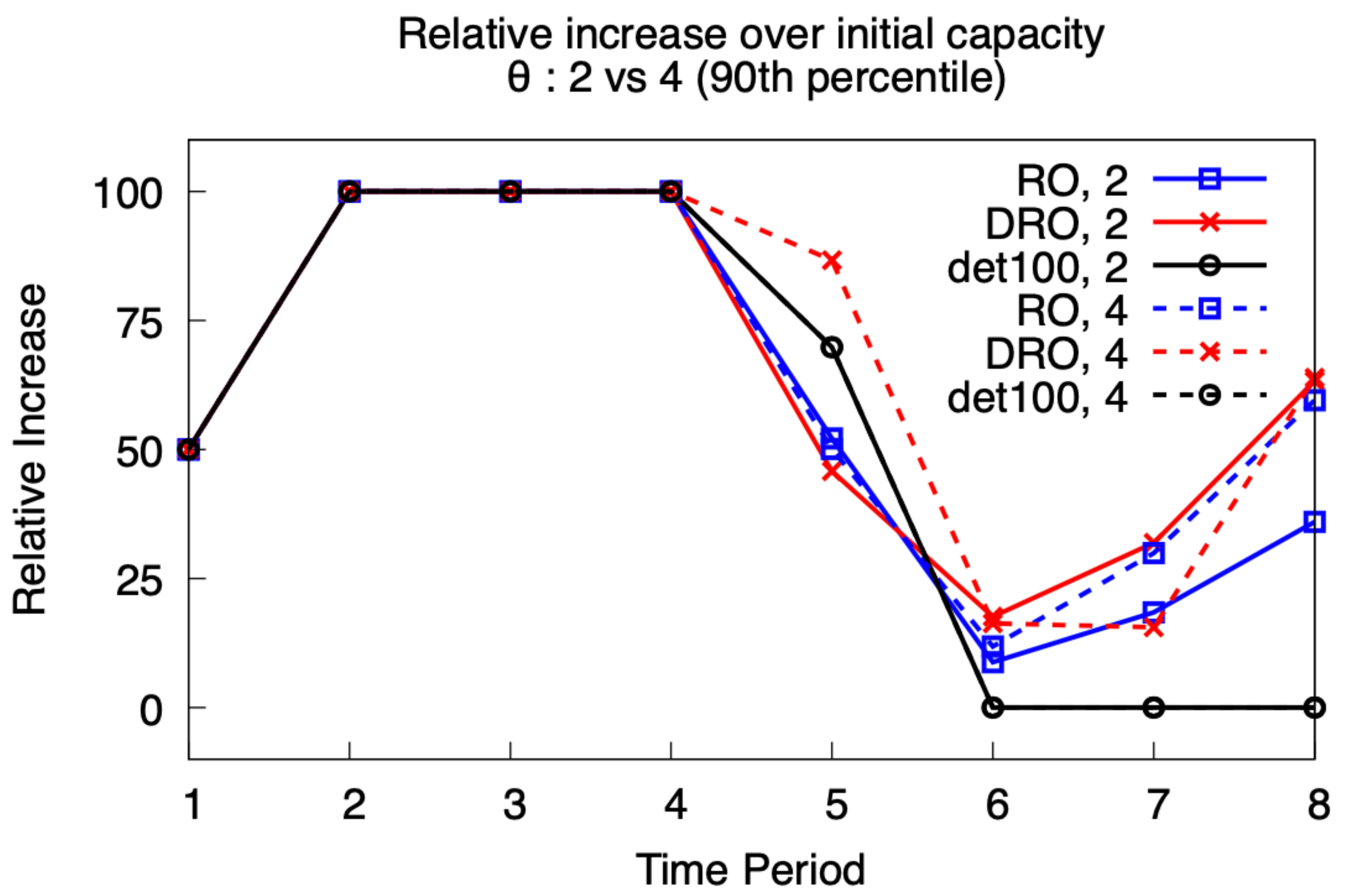
Structure of Expansion Policies





- All methods keep maximum capacity for the first three months (surge period).
- The det100 drops the most afterwards, whereas the other approaches maintain flexibility.
- The RO and DRO methods maintain more flexibility.

Sensitivity Analysis

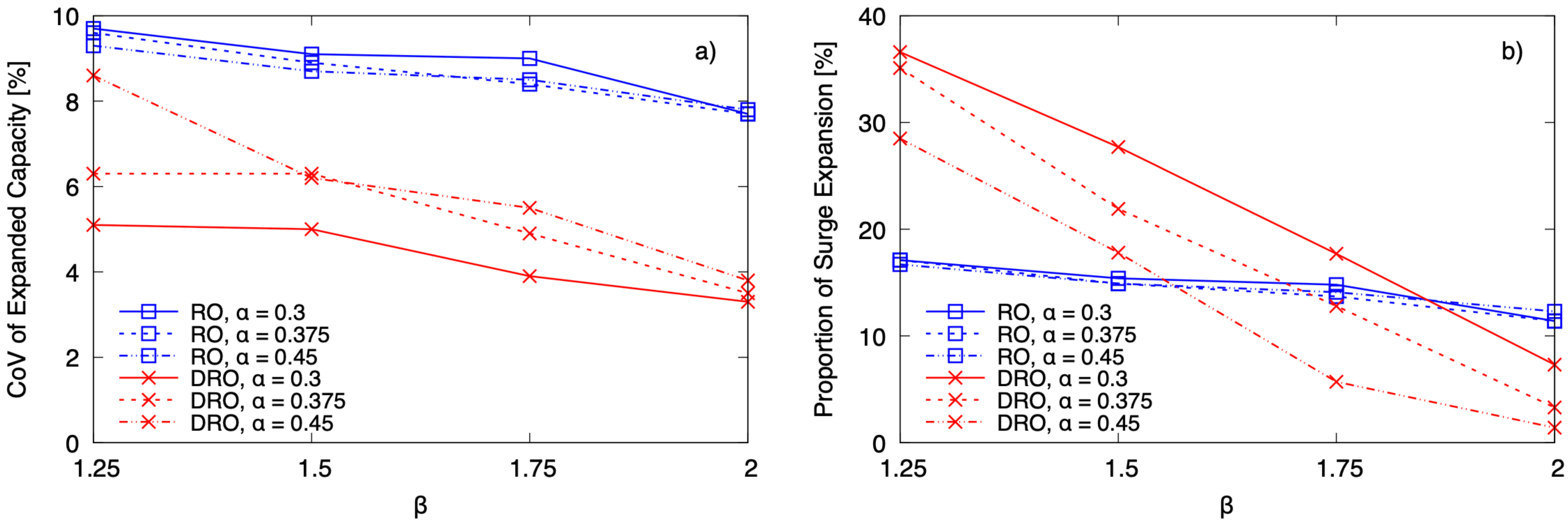
- Key Ratios

$$\alpha = \frac{\text{base expansion cost}}{\text{surgery cost}}$$

$$\beta = \frac{\text{expedited expansion cost}}{\text{base expansion cost}}$$

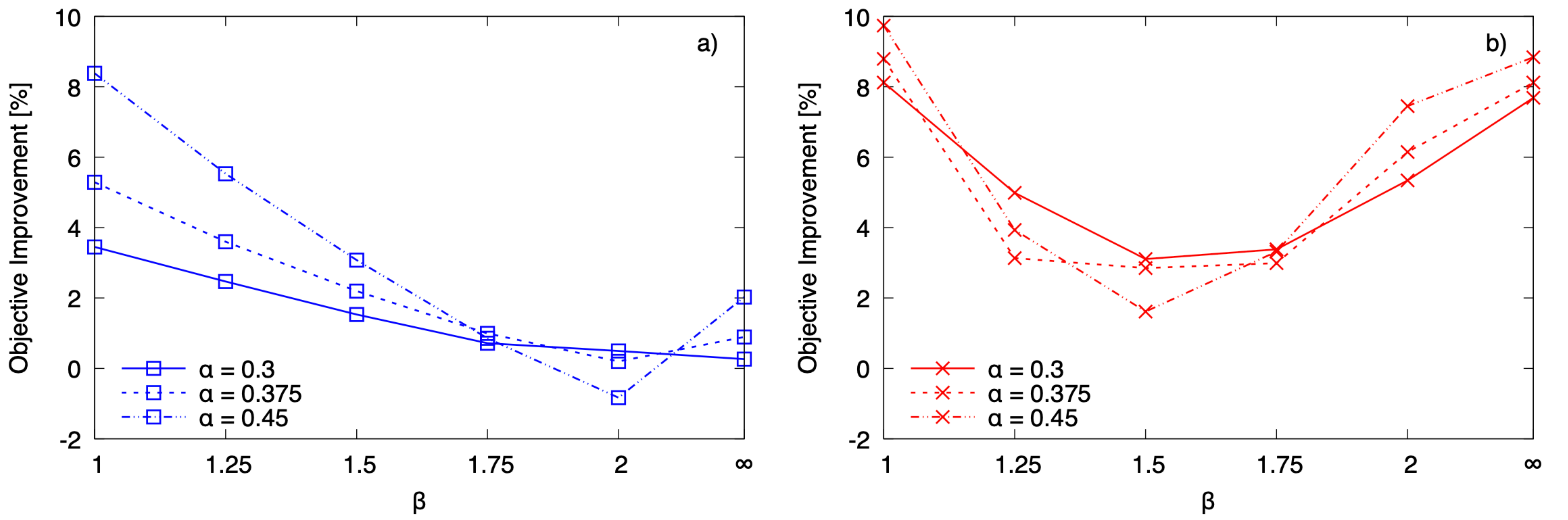
- α is the cost of base expansion as a fraction of the surgery cost
- β is an estimate of how much more expensive it is to do an expedited expansion than base expansion

Variation in Expanded Capacity



- Estimated from 100 scenarios
- RO is much less sensitive to α and decreases slightly with β
- For DRO, (i) CoV increases with α and decreases with β, (ii) Proportion of Surge Expansion decreases with α and β

Sensitivity of Objective



- Objective improvement (in percentage) over deterministic policies for RO (left) and DRO (right)
- If surge expansion is cheap or expensive then we get large objective improvements. For “cheap” because of adaptivity and for “expensive” because of the use of base expansion

Policy Comparison

Criteria	DRO	RO	Det100
Average performance	More effective	Less effective	benchmark
Performance under risky scenarios	Less effective	More effective	benchmark
Expansion structure	Slower cooldown; reserves higher capacity thereafter	Faster cooldown; reserves lower capacity thereafter	No adaptation
Utilization of surge expansion	Higher and sensitive to expansion costs	Lower and less sensitive to expansion costs	Never used
Impact of expansion costs on expansion structure	Sensitive	Less sensitive	No adaptation
Impact of expansion costs on objectives	Sensitive	Less sensitive	No adaptation

Conclusions

- Dynamic expansion of surgical capacity is necessary to clear a large number of deferred surgeries.
- Decision-making is challenging due to demand and departure uncertainty.
- Two optimization methods, based on RO and DRO, are developed.
- Proposed methods significantly improve objectives (5~10%) over deterministic policies on the hernia case study.
- Expansion structure and objective performance are analyzed and sensitivity analysis is performed.

Han E, Sharma K, Singh K, and Nohadani O. *Dynamic Capacity Management for Deferred Surgeries*. Under Review.

thank you!

Appendix: Example of Tree of Uncertainty Products

Theorem

If $\Xi = \times_{n=1}^{N} [0, \bar{\xi}_n]$ and the tree of uncertainty product satisfies that $k_i^* \neq k_j^*$ for any $i \neq j, i, j \in \mathcal{N}$, then (1) is equivalent to (2).

Example

The lifted set $\overline{\Xi}$ characterizes tight convex and concave envelopes of a function

$$\sum_{i=1}^{7} a_i\xi_i + b_1\xi_1\xi_2 + b_2\xi_1\xi_2\xi_3 + b_3\xi_1\xi_2\xi_4 + b_4\xi_1\xi_2\xi_4\xi_5 + b_5\xi_1\xi_2\xi_4\xi_6 + b_6\xi_1\xi_2\xi_4\xi_5\xi_7$$